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Substitution Example: $\sigma = [y/x, x+1/y, 5/z]$

We have
$dom(\sigma) = \{x, y, z\}$
$codom(\sigma) = \{x, y\}$.

Applying σ to the variables w and y gives

$w\sigma = w$
$y\sigma = x + 1$.

Notice in the second example the result is *not* $(y + 1)$. That is, the bindings in σ are not applied "one after the other", they are applied in parallel.

"Update" example: $\sigma[y \to x + 2] = [y/x, x + 2/y]$

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What substitutions will be needed for: suppose we want to refute the disequation $0 + s(0) \neq s(0)$ given the axiom $0 + x = x$.

Applying the substitution $[s(0)/x]$ to the axiom, that is,

$$(0 + x = x)[s(0)/x],$$

gives $0 + s(0) = s(0)$.

This equality replaces the left-hand-side of the disequation to produce $s(0) \neq s(0)$, which is obviously false.

In fact, the substitution above is a "unifier" (see later).

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Let $Var(t)$ be the set of variables occurring in term t.

Proposition For every term t, subsitution σ, and variable y,
if $y \notin Var(t)$ then $(t[y/z])[s/y] = t[s/z]$.

We shall denote the whole proposition above by q.

Example:

$$\begin{aligned}(f(z)[y/z])[s/y] &= f(y)[s/y] \\ &= f(s) \\ &= f(z)[s/z]\end{aligned}$$

Proof By structural induction.

1. Base case: $t = x$ for some $x \in X$.

1.1 $x = y$, then q holds trivially as $y \in Var(x)$.

1.2 $x \neq y$, then $y \notin Var(x) = x$.

If $x \neq z$ it follows

$$\begin{aligned}(x[y/z])[s/y] &= x[s/y] && \text{as } x \neq z \\ &= x && \text{as } x \neq y \\ &= x[s/z] && \text{as } x \neq z\end{aligned}$$

If $x = z$ it follows in a similar way

$$\begin{aligned}(z[y/z])[s/y] &= y[s/y] \\ &= s \\ &= z[s/z]\end{aligned}$$

2. Step case: let $t = f(t_1, \ldots, t_n) \in T_\Sigma(X)$ arbitrarily.
Assume q holds for all terms $t_1, \ldots, t_n$ (induction hypothesis).
Assume that $y \notin Var(t)$ (otherwise the step case follows trivially).
It follows trivially $y \notin Var(t_i)$ for all $i = 1, \ldots, n$.

By the induction hypothesis

$$(t_i[y/z])[s/y] = t_i[s/z] \qquad (*)$$

Now,

$$\begin{aligned}(f(t_1, \ldots, t_n)[y/z])[s/y] &= f((t_1[y/z])[s/y], \ldots, (t_n[y/z])[s/y]) && \text{by homomorphic extension} \\ &= f(t_1[s/z], \ldots, t_n[s/z] && \text{by (*)} \\ &= f(t_1, \ldots, t_n)[s/z] && \text{by homomorphic extension}\end{aligned}$$

This completes the proof. □

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Let $\mathcal{F} = \exists x \; P(x, z)$.

Is F valid? Is F satisfiable?

It holds that F is satisfiable. Take *e.g.*

$$\mathcal{A} = (\mathbb{N}, \emptyset, P : \{(0,1)\}).$$

It follows $\mathcal{A}, [z \to 1] \models F$.

But we also have $\mathcal{A}, [z \to 0] \not\models F$, hence F is not valid.

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Proposition F entails G iff $(F \to G)$ is valid.

That is, entailment between two formulas is equivalent to the validity of the corresponding implication.

Proof By expanding defintions:

	F entails G
iff	(for all $\mathcal{A} \in \Sigma$-Alg and $\beta \in X \to U_\mathcal{A}$)
	if $\mathcal{A}, \beta \models F$ then $\mathcal{A}, \beta \models G$
iff	if $\mathcal{A}(\beta)(F) = 1$ then $\mathcal{A}(\beta)(G) = 1$
iff	$\mathcal{A}(\beta)(F \to G) = 1$
iff	$\mathcal{A}, \beta \models F \to G$
iff	$F \to G$ is valid. □

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Consider the entailment

$$\exists x P(x) \land \forall x(P(x) \to \exists y Q(y)) \models \exists z Q(z)$$

By Proposition 2.4, this entailment holds if and only if the formula

$$\underline{\exists x P(x)} \land \forall x(P(x) \to \exists y Q(y)) \land \underline{\neg \exists z Q(z)}$$

is unsatisfiable.

We are going to transform the formula into Prenex Normal Form. Recall that pulling quantifiers outside involves renaming the bound variables by fresh ones. As a compromise, for better readability, we do this only if necesssary, i.e., when variables would otherwise be bound unintentionally.

Working on the underlined subformulas above gives

$$\exists x(P(x) \land \underline{\forall x \exists y(P(x) \to Q(y))} \land \forall z \neg Q(z)).$$

Now, we pull out the quantifiers in the underlined subformula. Notice that this time the variable x must be renamed. Let us chose $[w/x]$ for that, and so we get

$$\exists x \forall w \exists y \forall z(P(x) \land (P(w) \to Q(y)) \land \neg Q(z)).$$

This formula is in Prenex Normal Form. The next step is Skolemization, to remove (all) existential quantifiers. We proceed from the left to right, i.e., outermost existential quantifiers are removed first:

For $\exists x$, pick $[a/x]$, where a is a fresh constant, giving

$$\forall w \exists y \forall z(P(a) \land (P(w) \to Q(y)) \land \neg Q(z)).$$

For $\exists y$, which occurs in the scope of $\forall w$, pick $[f(w)/y]$, where f is a fresh unary function symbol, giving

$$\forall w \forall z(P(a) \land (P(w) \to Q(f(w))) \land \neg Q(z)).$$

Finally, removing the universal quantifiers, transforming the resulting formula into CNF and writing it as a set gives the Clause Normal Form

$$\{P(a), \neg P(w) \lor Q(f(w)), \neg Q(z)\}.$$

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$U_{\mathcal{A}}$: The "Herband Universe".

Example: $\Sigma_{\mathcal{A}} = (\{0/0, s/1, +/2\}, \{< /2, \leq /2\})$

Then $U_{\mathcal{A}} = \{0, s(0), s(s(0)), \ldots$
$0 + 0, 0 + s(0), 0 + s(s(0)), \ldots$
$s(0) + 0, s(s(0)) + 0, \ldots$
$\ldots\}$

That is, the set of all ground terms.

Consequences:

- Assignments are nothing but substitutions with empty codomain.
- Interpretation function maps every term to "itself"

  e.g. $\beta = [x \to s(0)] \mathrel{\hat{=}} [s(0)/x]$

  $\mathcal{A}(\beta)(x + s(0)) = s(0) + s(0)$

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E.g. For a suitable $\mathcal{A} : \mathcal{A} \models x > 0 \vee \neg(x > 0)$ (*note*: "$\forall x$" implicit)

but neither $\mathcal{A} \models x > 0$ nor $\mathcal{A} \models \neg(x > 0)$.

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Example: let $a > b > c > d$. Then

$S_1 = \{a, a, a, b, c\}$
$\succ_{\text{mul}}$
$S_2 = \{a, a, b, b, b, d\}$

Alternatively: $S_1 \succ_{\text{mul}} S_2$ iff
S_2 can be obtained by replacing some (at least one) element(s) in S_1 by (zero or more) smaller elements.

Example: in S_1, replacing the third a by two b's, and the last c by d gives S_2, with the picture:

```
{a, a, a, b, c}
        |\    \
{a, a, b, b, b, d}
```

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Resolution inference example:

$$\frac{R(y) \vee P(y) \qquad Q(x) \vee \neg P(f(x))}{R(f(x)) \vee Q(x)}$$

where $\sigma = \text{mgu}(P(y), P(f(x))) = [f(x)/y]$.

Notice that the two clauses in the premise are variable disjoint.

However, the resolution inference rule is not applicable with the premises $R(x) \vee P(x)$ and $Q(x) \vee \neg P(f(x))$, as these clauses are not variable disjoint; the substitution $\sigma = [f(x)/x]$ is not a unifier for $P(x)$ and $P(f(x))$.

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Unification example 1 (“rule (i)” refers to the rule on line i in “Rule Based Naive Standard Unific-ation”):

$$\begin{aligned} E_1 = \quad & f(x,x) \doteq f(g(y),z) & \\ \Rightarrow_{SU} \quad & x \doteq g(y),\ x \doteq z & \text{(by rule (2))} \\ \Rightarrow_{SU} \quad & x \doteq g(y),\ g(y) \doteq z & \text{(by rule (4))} \\ \Rightarrow_{SU} \quad & x \doteq g(y),\ z \doteq g(y) & \text{(by rule (6))} \end{aligned}$$

No more rule is applicable to the set in the last line. It follows $\mathrm{mgu}(E_1) = [g(y)/x,\ g(y)/z]$.

Unification example 2:

$$\begin{aligned} E_2 = \quad & f(f(x)) \doteq f(x) & \\ \Rightarrow_{SU} \quad & f(x) \doteq x & \text{(by rule (2))} \\ \Rightarrow_{SU} \quad & x \doteq f(x) & \text{(by rule (6))} \\ \Rightarrow_{SU} \quad & \bot & \text{(by rule (5))} \end{aligned}$$

The result $\bot$ indicates that E_2 does not have a unifier.