Instance Based Methods

TABLEAUX 2005 Tutorial (Koblenz, September 2005)

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Purpose of Tutorial

Instance Based Methods (IMs): a family of calculi and proof procedures for first-order clause logic, developed during past ten years

Tutorial provides overview about the following

- **Common principles behind IMs, some calculi, proof procedures**
- **Comparison among IMs, difference from tableaux and resolution**
- **Ranges of applicability/non-applicability**
- **Improvements and extensions: universal variables, equality, . . .**
- **Picking up SAT techniques**
- **Implementations and implementation techniques**

Setting the Stage

Skolem-Herbrand-Löwenheim Theorem

$\forall\phi$ is unsatisfiable iff some finite set of ground instances $\{\phi\gamma_1, \ldots, \phi\gamma_n\}$ is unsatisfiable

For refutational theorem proving (i.e. start with negated conjecture) it thus suffices to

- enumerate growing finite sets of such ground instances, and
- test each for propositional unsatisfiability. Stop with “unsatisfiable” when the first propositionally unsatisfiability set arrives

This has been known for a long time: Gilmore’s algorithm, DPLL

It is also a common principle behind IMs

Setting the Stage

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It is also a common principle behind IMs

So what's special about IMs? Do this in a clever way!

An early IM: the DPLL Procedure

Preprocessing:

Given Formula		Clause Form
$\forall x\, \exists y\, P(y,x)$ $\wedge \forall z\, \neg P(z,a)$	$\longrightarrow$	$P(f(x),x)$ $\neg P(z,a)$

Outer loop:
Grounding

Inner loop:
Propositional
DPLL

An early IM: the DPLL Procedure

Given Formula → **Clause Form**

Preprocessing:

$\forall x\, \exists y\, P(y, x)$
$\land \forall z\, \neg P(z, a)$

$\longrightarrow$

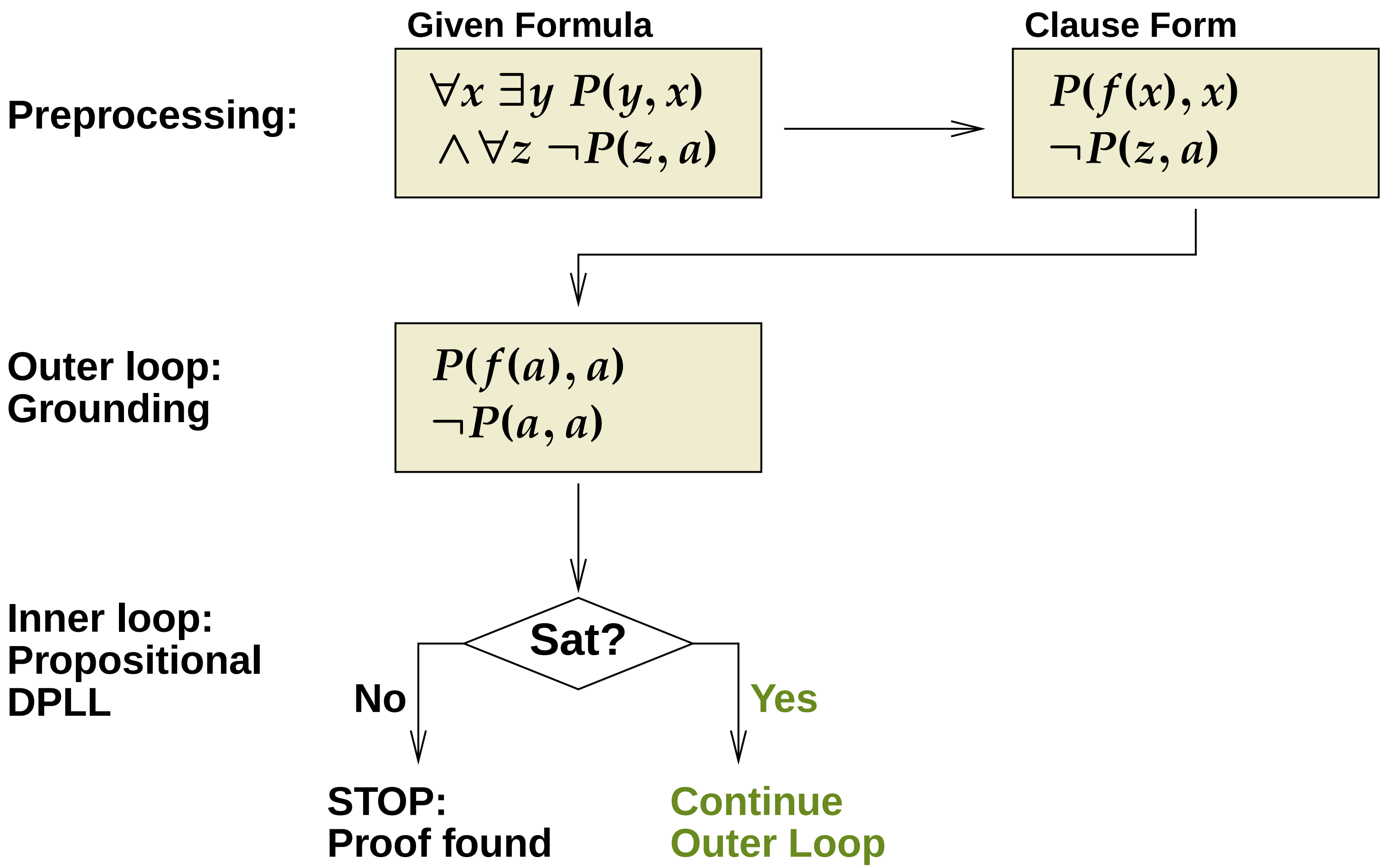
$P(f(x), x)$
$\neg P(z, a)$

Outer loop: Grounding

$P(f(a), a)$
$\neg P(a, a)$

Inner loop: Propositional DPLL

An early IM: the DPLL Procedure



An early IM: the DPLL Procedure

	Given Formula		Clause Form
Preprocessing:	$\forall x \, \exists y \, P(y, x)$ $\land \forall z \, \neg P(z, a)$	$\longrightarrow$	$P(f(x), x)$ $\neg P(z, a)$
Outer loop: Grounding	$P(f(a), a)$ $\neg P(a, a)$	. . . .	$P(f(a), a)$ $\neg P(a, a)$ $\neg P(f(a), a)$

Inner loop: Propositional DPLL

An early IM: the DPLL Procedure

Preprocessing:

Given Formula

$\forall x\, \exists y\, P(y, x)$
$\wedge \forall z\, \neg P(z, a)$

Clause Form

$P(f(x), x)$
$\neg P(z, a)$

Outer loop:
Grounding

$P(f(a), a)$
$\neg P(a, a)$

$P(f(a), a)$
$\neg P(a, a)$
$\neg P(f(a), a)$

Inner loop:
Propositional
DPLL

Sat?

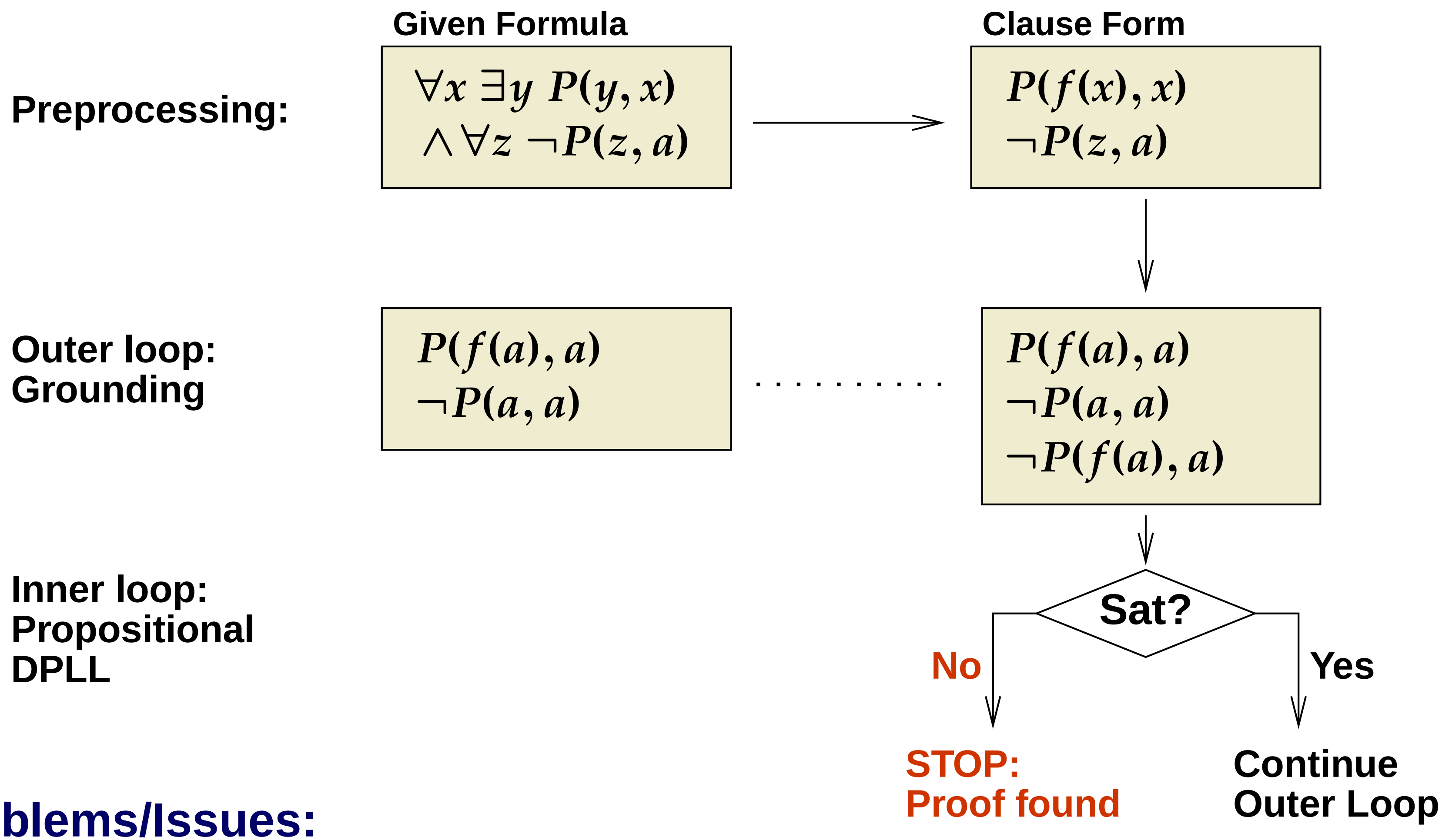
No

STOP:
Proof found

Yes

Continue
Outer Loop

An early IM: the DPLL Procedure



Problems/Issues:

- Controlling the grounding process in *outer loop* (irrelevant instances)
- Repeat work *across* inner loops
- Weak redundancy criterion *within* inner loop

Part I: Overview of IMs

- Classification of IMs and some representative calculi
- Emphasis not too much on the details
- We try to work out common principles and also differences
- Comparison with Resolution and Tableaux
- Applicability/Non-Applicability

Development of IMs (I)

Purpose of this slide

- List existing methods (apologies for "forgotten" ones . . .)
- Define abbreviations used later on
- Provide pointer to literature
- Itemize structure indicates reference relation (when obvious)
- *Not:* table of contents of what follows
  (presentation is systematic instead of historical)

DPLL – Davis-Putnam-Logemann-Loveland procedure

[Davis and Putnam, 1960], [Davis *et al.*, 1962b], [Davis *et al.*, 1962a], [Davis, 1963], [Chinlund *et al.*, 1964]

- FDPLL – First-Order DPLL [Baumgartner, 2000]
  - ME – Model Evolution Calculus [Baumgartner and Tinelli, 2003]
    - ME with Equality [Baumgartner and Tinelli, 2005]

Development of IMs (II)

HL – Hyperlinking [Lee and Plaisted, 1992]

- SHL – Semantic Hyper Linking [Chu and Plaisted, 1994]
  - OSHL – Ordered Semantic Hyper Linking [Plaisted and Zhu, 1997]

PPI – Primal Partial Instantiation (1994) [Hooker *et al.*, 2002]

- "Inst-Gen" [Ganzinger and Korovin, 2003]

MACE-Style Finite Model Buiding [McCune, 1994],. . . , [Claessen and Sörensson, 2003]

DC – Disconnection Method [Billon, 1996]

- HTNG - Hyper Tableaux Next Generation [Baumgartner, 1998]
- DCTP – Disconnection Tableaux [Letz and Stenz, 2001]

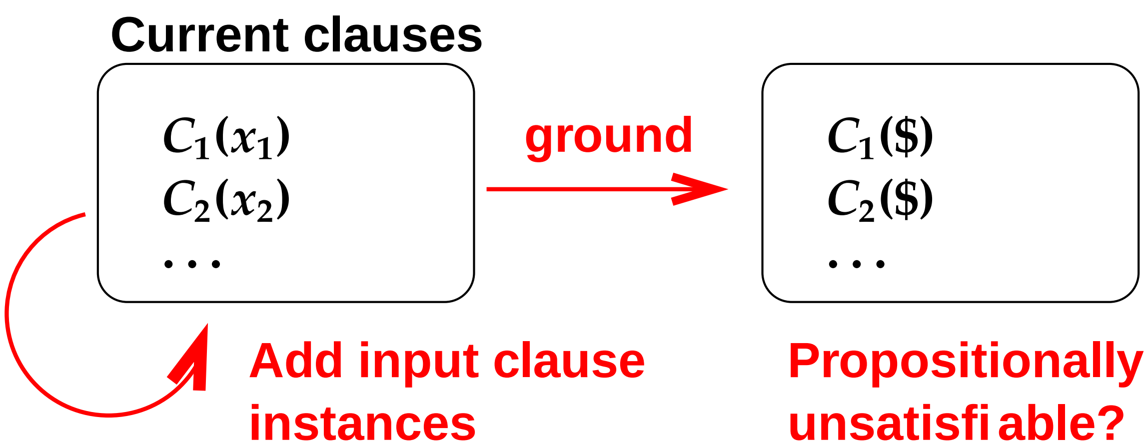
Ginsberg & Parkes method [Ginsberg and Parkes, 2000]

OSHT – Ordered Semantic Hyper Tableaux [Yahya and Plaisted, 2002]

Two-Level vs. One-Level Calculi

Two-Level Calculi

- Separation between instance generation and SAT solving phase
- Uses (arbitrary) propositional SAT solver as a subroutine
- DPLL, HL, SHL, OSHL, PPI, Inst-Gen
- Problem: how to tell SAT solver e.g. $\forall x P(x)$?

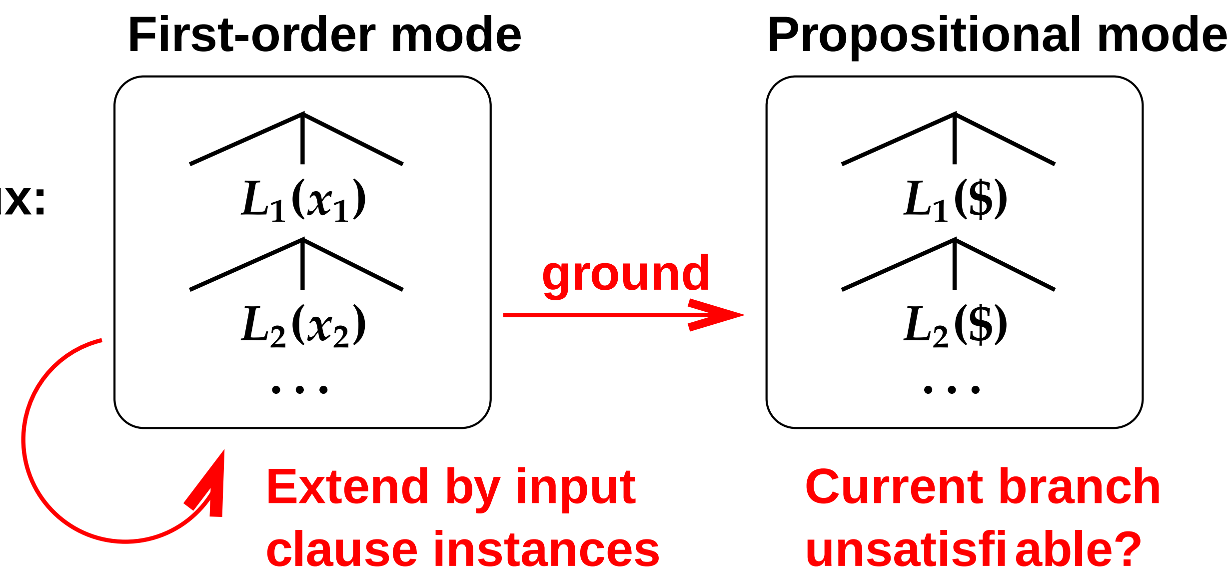


Two-Level vs. One-Level Calculi

One-Level Calculi

- Monolithic: one single base calculus, two modes of operation
- First-order mode: builds base calculus data structure from input clause instances
- Propositional mode: $\$$-instance of data structures drives first-order mode
- HyperTableaux NG, DCTP (see Part II), OSHT, FDPLL, ME

E.g. Tableaux:



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- **Monolithic: one single base calculus, two modes of operation**
- **First-order mode: builds base calculus data structure from input clause instances**
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- **HyperTableaux NG, DCTP (see Part II), OSHT, FDPLL, ME**

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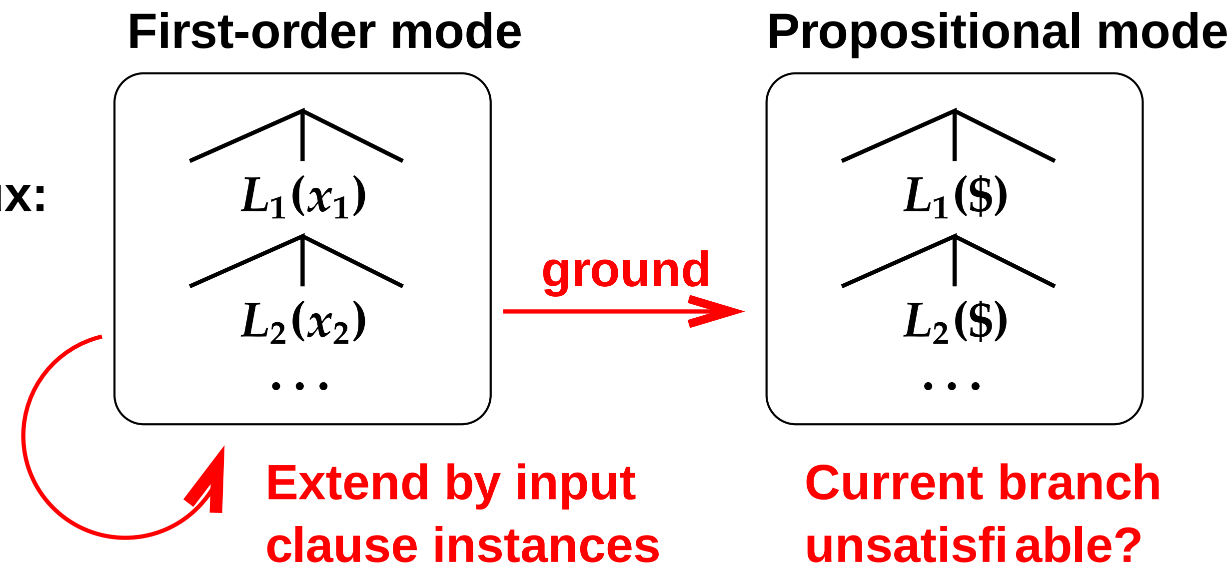

Next: two-level calculus "Inst-Gen"

Inst-Gen

- We have chosen Inst-Gen for presentation because of its elegance and simplicity
- Talk proceeds with
  - Idea behind Inst-Gen

    (it provides a clue to the working of two-level calculi)
  - Inst-Gen calculus
  - Comparison to Resolution
  - Mentioning some improvements, as justified by “idea behind”
- See [Ganzinger and Korovin, 2003] for details

Inst-Gen - Underlying Idea (I)

Important notation: $\perp$ **denotes both a unique constant and a substitution that maps every variable to** $\perp$**.**

Example (S **is "current clause set"):**

$$S: \quad P(x,y) \vee P(y,x) \qquad\qquad S\perp: \quad P(\perp,\perp) \vee P(\perp,\perp)$$

$$\neg P(x,x) \qquad\qquad \neg P(\perp,\perp)$$

Analyze $S\perp$**:**

Case 1: SAT detects unsatisfiability of $S\perp$

Then Conclude S **is unsatisfiable**

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Analyze $S\perp$**:**

Case 1: SAT detects unsatisfiability of $S\perp$

Then Conclude S **is unsatisfiable**

But what if $S\perp$ **is satisfied by some model, denoted by** $I_\perp$**?**

Inst-Gen - Underlying Idea (II)

Main idea: associate to model $I_\perp$ of $S\perp$ a **candidate model** I_S of S.

Calculus goal: add instances to S so that I_S becomes a model of S

Example:

$$S: \quad \underline{P(x)} \vee Q(x) \qquad\qquad S\perp: \quad \underline{P(\perp)} \vee Q(\perp)$$

$$\underline{\neg P(a)} \qquad\qquad\qquad\qquad \underline{\neg P(a)}$$

Analyze $S\perp$**:**

Case 2: SAT detects model $I_\perp = \{P(\perp), \neg P(a)\}$ of $S\perp$

Case 2.1: candidate model $I_S = \{\neg P(a)\}$ derived from

literals <u>selected</u> in S by $I_\perp$ is not a model of S

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literals <u>selected</u> in S by $I_\perp$ is not a model of S

Add "problematic" instance $P(a) \vee Q(a)$ to S to refine I_S

Inst-Gen - Underlying Idea (III)

Clause set after adding $P(a) \vee Q(a)$

$$S: \quad \underline{P(x)} \vee Q(x) \qquad\qquad S\bot: \quad \underline{P(\bot)} \vee Q(\bot)$$

$$P(a) \vee \underline{Q(a)} \qquad\qquad P(a) \vee \underline{Q(a)}$$

$$\underline{\neg P(a)} \qquad\qquad \underline{\neg P(a)}$$

Analyze $S\bot$:

Case 2: SAT detects model $I_\bot = \{P(\bot), Q(a), \neg P(a)\}$ of $S\bot$

Case 2.2: candidate model $I_S = \{Q(a), \neg P(a)\}$ derived from

literals <u>selected</u> in S by $I_\bot$ *is* a model of S

Then conclude S is satisfiable

Inst-Gen - Underlying Idea (III)

Clause set after adding $P(a) \vee Q(a)$

$S:$	$\underline{P(x)} \vee Q(x)$	$S\bot:$	$\underline{P(\bot)} \vee Q(\bot)$
	$P(a) \vee \underline{Q(a)}$		$P(a) \vee \underline{Q(a)}$
	$\underline{\neg P(a)}$		$\underline{\neg P(a)}$

Analyze $S\bot$:

Case 2: SAT detects model $I_\bot = \{P(\bot), Q(a), \neg P(a)\}$ of $S\bot$

Case 2.2: candidate model $I_S = \{Q(a), \neg P(a)\}$ derived from

literals <u>selected</u> in S by $I_\bot$ *is* a model of S

Then conclude S is satisfiable

How to derive candidate model I_S?

Inst-Gen - Model Construction

It provides (partial) interpretation for S_{ground} for given clause set S

$$S: \quad \underline{P(x)} \vee Q(x) \qquad\qquad \Sigma = \{a, b\},\ S_{\text{ground}}: \quad \underline{P(b)} \vee Q(b)$$

$$P(a) \vee \underline{Q(a)} \qquad\qquad P(a) \vee \underline{Q(a)}$$

$$\underline{\neg P(a)} \qquad\qquad \underline{\neg P(a)}$$

- **For each $C_{\text{ground}} \in S_{\text{ground}}$ find most specific $C \in S$ that can be instantiated to C_{ground}**
- **Select literal in C_{ground} corresponding to selected literal in that C**
- **Add selected literal of that C_{ground} to I_S if not in conflict with I_S**

Thus, $I_S = \{P(b), Q(a), \neg P(a)\}$

Inst-Gen - Summary so far

- **Previous slides showed the main ideas underlying the working of calculus - not the calculus itself**
- **The models $I_\perp$ and the candidate model I_S are not needed in the calculus, but justify improvements**
- **And they provide the conceptual tool for the completeness proof: as instances of clauses are added, the initial approximation of a model of S is refined more and more**
- **The purpose of this refinement is to remove conflicts "A – $\neg A$" by selecting different literals in instances of clauses**
- **If this process does not lead to a refutation, every ground instance $C\gamma$ of a clause $C \in S$ will be assigned true by some sufficiently developed candidate model**

Inst-Gen Inference Rule

$$\text{Inst-Gen}\ \frac{C \vee L \qquad \overline{L'} \vee D}{(C \vee L)\theta \quad (\overline{L'} \vee D)\theta} \quad \text{where}$$

(i) $\theta = \text{mgu}(L, L')$, and

(ii) θ is a **proper instantiator:** maps some variables to nonvariable terms

Example:

$$\text{Inst-Gen}\ \frac{Q(x) \vee P(x, b) \quad \neg P(a, y) \vee R(y)}{Q(a) \vee P(a, b) \quad \neg P(a, b) \vee R(b)} \quad \text{where}$$

(i) $\theta = \text{mgu}(P(x, b), \neg P(a, y)) = \{x \longrightarrow a, y \longrightarrow b\}$, and

(ii) θ is a proper instantiator

Inst-Gen - Outer Loop

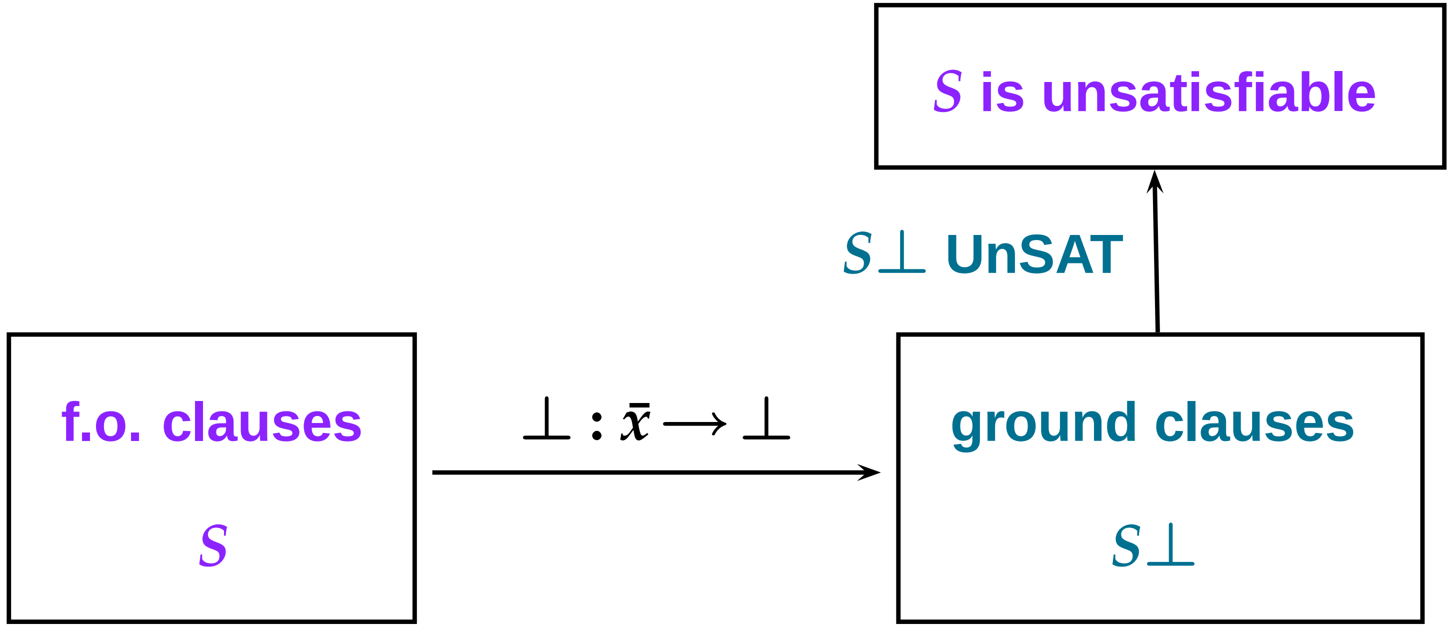
f.o. clauses

S

Inst-Gen - Outer Loop

$$\boxed{\begin{array}{c}\text{f.o. clauses}\\ S\end{array}} \xrightarrow{\perp : \bar{x} \to \perp} \boxed{\begin{array}{c}\text{ground clauses}\\ S\perp\end{array}}$$

Inst-Gen - Outer Loop



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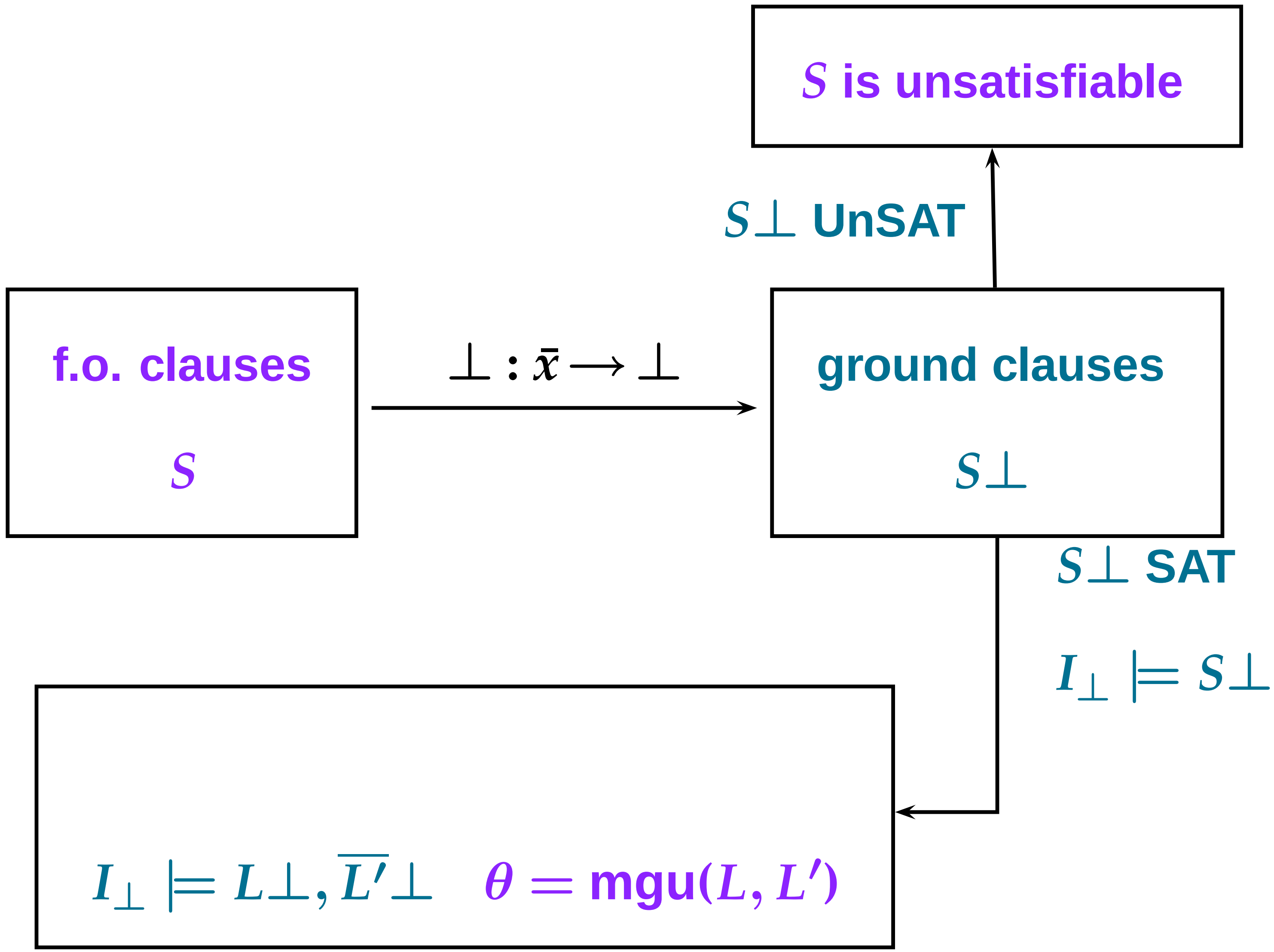

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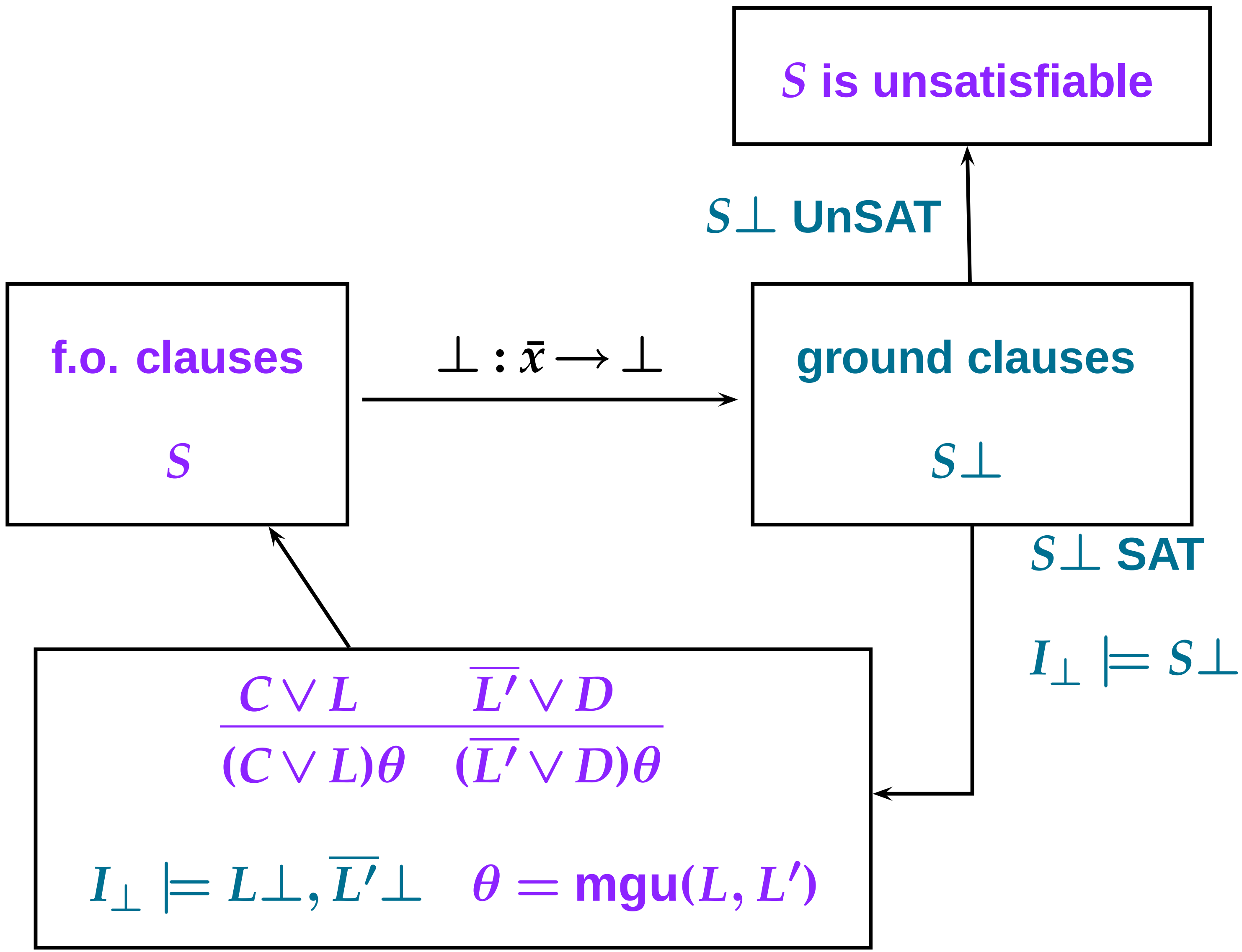

Properties and Improvements

- As efficient as possible in propositional case
- <u>Literal selection</u> *in the calculus*
  - Require "back channel" from SAT solver (output of models) to select literals in S (as obtained in $I_\perp$)
  - Restrict inference rule application to selected literals
  - Need only consider instances falsified in I_S
  - Allows to extract model if S is finitely saturated
  - Flexibility: may change models $I_\perp$ arbitrarily during derivation
- Hyper-type inference rule, similar to Hyper Linking [Lee and Plaisted, 1992]
- Subsumption deletion by proper subclauses
- Special variables: allows to replace SAT solver by solver for richer fragment (guarded fragment, two-variable fragment)

Resolution vs. Inst-Gen

Resolution

$$\frac{(C \vee L) \qquad (\overline{L'} \vee D)}{(C \vee D)\theta}$$

$$\theta = \mathrm{mgu}(L, L')$$

- Inefficient in propositional case
- Length of clauses can grow fast
- Recombination of clauses
- Subsumption deletion
- A-Ordered resolution: selection based on term orderings
- Difficult to extract model
- Decides guarded fragment, two-variable fragment, some classes defined by Leitsch et al., not Bernays-Schönfinkel class

Inst-Gen

$$\frac{C \vee L \qquad \overline{L'} \vee D}{(C \vee L)\theta \quad (\overline{L'} \vee D)\theta}$$

$$\theta = \mathrm{mgu}(L, L')$$

- Efficient in propositional case
- Length of clauses fixed
- No recombination of clauses
- Subsumption deletion limited
- Selection based on propositional model
- Easy to extract model
- Decides Bernays-Schönfinkel class, nothing else known yet
- Current CASC-winning provers use Resoluti

Other Two-Level Calculi (I)

DPLL - Davis-Putnam-Logemann-Loveland Procedure

- Weak concept of redundancy already present (purity deletion)

PPI – Primal Partial Instantiation

- Comparable to Inst-Gen, but see [Jacobs and Waldmann, 2005]
- With fixed iterative deepening over term-depth bound

MACE-Style Finite Model Buiding (Different Focus)

- Enumerate finite domains $\{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots$
- Transform clause set to encode search for model with finite domain
- Apply (incremental) SAT solver
- Complete for finite models, not refutationally complete

Other Two-Level Calculi (II) - HL and SHL

HL - Hyper Linking (Clause Linking)

- Uses hyper type of inference rule, based on simultaneous mgu of nucleus and electrons
- Doesn't use selection (no guidance from propositional model)

SHL - Semantic Hyper Linking

- Uses "back channel" from SAT solver to guide search: find *single* ground clause $C\gamma$ so that $I_\perp \not\models C\gamma$ and add it
- Doesn't use unification; basically guess ground instance, but ...
- Practical effectiveness achieved by other devices:
  - Start with "natural" initial interpretation
  - "Rough resolution" to eliminate "large" literals
  - Predicate replacement to unfold definitions [Lee and Plaisted, 1989]
- See also important paper [Plaisted, 1994]

Other Two-Level Calculi (III) - OSHL

OSHL - Ordered Semantic Hyper Linking [Plaisted and Zhu, 1997], [Plaisted and Zhu, 2000]

- Goal-orientation by chosing "natural" initial interpretation I_0 that falsifies (negated) theorem clause, but satisfies most of the theory clauses
- Stepwisely modify I_0
  Modified interpretation represented as $I_0(L_1, \ldots, L_m)$
  (which is like I_0 except for ground literals $L_1, \ldots, L_m$)
- Completeness via fair enumeration of modifications
- Special treatment of unit clauses
- Subsumption by proper subclauses
- Uses A-ordered resolution as propositional decision procedure

OSHL Proof Procedure

Input: S, I_0 ;; S input clauses, I_0 initial interpretation
I := I_0 ;; Current interpretation
G := $\{\}$;; Set of current ground instances of clauses of S
while $\{\} \notin G$ do
    if $I \models S$;; ... and this can be detected
        then return "satisfiable"
    search $C \in S$ and γ
        such that $I \not\models C\gamma$;; Instance generation
    G := $simplify(G, C\gamma)$;; Have $C\gamma \in G$ after simplification
    I := $update(I_0, G)$;; Update such that $I \models G$
od
return "unsatisfiable"

How to search C and γ for given $I = I_0(L_1, \ldots, L_m)$

- Guess $C \in S$ and partition $C = C_1 \cup C_2$
- Let θ matcher of C_1 to $(\overline{L_1}, \ldots, \overline{L_m})$
- Guess δ s.th. $I_0(L_1, \ldots, L_m) \not\models C\gamma$, where $\gamma = \theta\delta$

Search and Update in OSHL

$I_o = \{Ra\}$

(all other atoms false)

S:

(1) $R(a) \leftarrow$

(2) $P(x) \leftarrow R(a)$

(3) $R(y) \vee Q(x,y) \leftarrow P(x)$

(4) $\leftarrow Q(a,c)$

(5) $\leftarrow R(c)$

OSHL Refutation:

(2)	I_0	$\not\models$	$P(x) \leftarrow R(a)$
	I_0	$\not\models$	$P(a) \leftarrow R(a)$
(3)	$I_0(P(a))$	$\not\models$	$R(y) \vee Q(x,y) \leftarrow P(x)$
	$I_0(P(a))$	$\not\models$	$R(y) \vee Q(a,y) \leftarrow P(a)$
	$I_0(P(a))$	$\not\models$	$R(c) \vee Q(a,c) \leftarrow P(a)$
(5)	$I_0(P(a), R(c))$	$\not\models$	$\leftarrow R(c)$
(4)	$I_0(P(a), Q(a,c))$	$\not\models$	$\leftarrow Q(a,c)$
(1)	$I_0(\neg R(a))$	$\not\models$	$R(a) \leftarrow$

Unsatisfiable

IMs - Classification

Recall:

- **Two-level calculi: instance generation separated from SAT solving – may use any SAT solver**
- **One-level calculi: monolithic, with two modes of operation: First-order mode and propositional mode**
  Developed so far:

IM	Extended Calculus
DC	Connection Method, Tableaux
DCTP	Tableaux
OSHT	Hyper Tableaux
Hyper Tableaux NG	Hyper Tableaux
FDPLL	DPLL
ME	DPLL

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Next: one-level calculus: FDPLL (simpler) / ME (better)

Motivation for FDPLL/ME

F DPLL: lifting of propositional core of DPLL to *F* irst-order logic

Why?

- Migrate to the first-order level those very effective techniques developed for propositional DPLL
- From propositional DPLL: binary splitting, backjumping, learning, restarts, selection heuristics, simplification, . . .
  Not all achieved yet; simplification not in FDPLL, but in ME
- Successful first-order techniques: unification, special treatment of unit clauses, subsumption (limited)
- *Theorem Proving:* alternative to established methods
- *Model computation:*
  counterexamples, diagnosis, abduction, planning, nonmonotonic reasoning,. . . – largely unexplored

Contents FDPLL/ME Part

- Propositional DPLL as a semantic tree method
- FDPLL calculus
- Model Evolution calculus
- FDPLL/ME vs. OSHL
- FDPLL/ME vs. Inst-Gen

Propositional DPLL as a Semantic Tree Method

(1) $A \vee B$ (2) $C \vee \neg A$ (3) $D \vee \neg C \vee \neg A$ (4) $\neg D \vee \neg B$

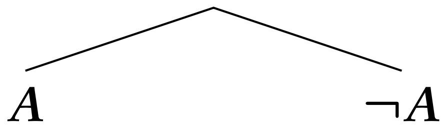
$\langle$**empty tree**$\rangle$

$$\{\} \not\models A \vee B$$
$$\{\} \models C \vee \neg A$$
$$\{\} \models D \vee \neg C \vee \neg A$$
$$\{\} \models \neg D \vee \neg B$$

- **A Branch stands for an interpretation**
- ***Purpose of splitting:*** **satisfy a clause that is currently falsified**
- **Close branch if some clause is plainly falsified by it ($\star$)**

Propositional DPLL as a Semantic Tree Method

(1) $A \lor B$ **(2)** $C \lor \neg A$ **(3)** $D \lor \neg C \lor \neg A$ **(4)** $\neg D \lor \neg B$



$$\{A\} \models A \lor B$$
$$\{A\} \not\models C \lor \neg A$$
$$\{A\} \models D \lor \neg C \lor \neg A$$
$$\{A\} \models \neg D \lor \neg B$$

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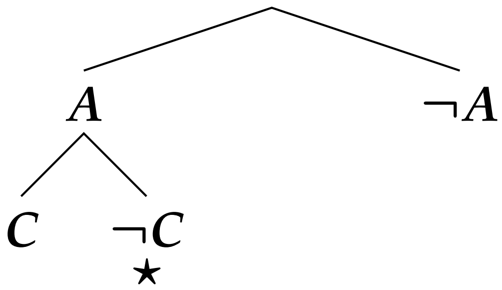
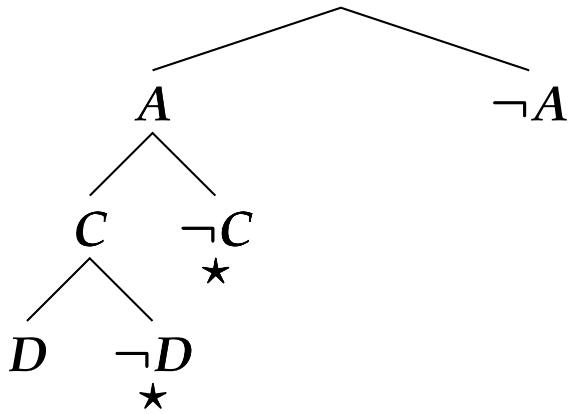


$$\{A, C\} \models A \vee B$$
$$\{A, C\} \models C \vee \neg A$$
$$\{A, C\} \not\models D \vee \neg C \vee \neg A$$
$$\{A, C\} \models \neg D \vee \neg B$$

- A Branch stands for an interpretation
- *Purpose of splitting:* satisfy a clause that is currently falsified
- Close branch if some clause is plainly falsified by it ($\star$)

Propositional DPLL as a Semantic Tree Method

(1) $A \vee B$ **(2)** $C \vee \neg A$ **(3)** $D \vee \neg C \vee \neg A$ **(4)** $\neg D \vee \neg B$



$\{A, C, D\} \models A \vee B$

$\{A, C, D\} \models C \vee \neg A$

$\{A, C, D\} \models D \vee \neg C \vee \neg A$

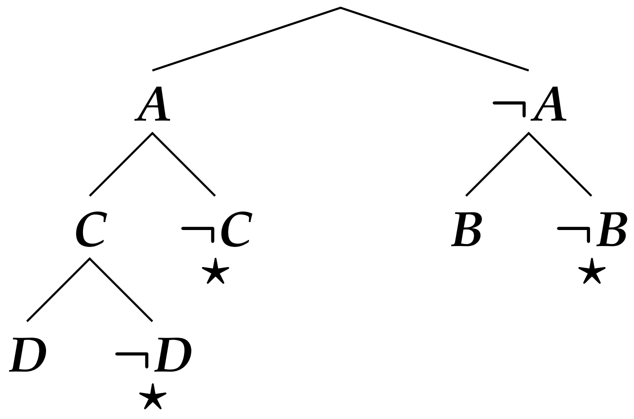
$\{A, C, D\} \models \neg D \vee \neg B$

Model $\{A, C, D\}$ found.

- **A Branch stands for an interpretation**
- ***Purpose of splitting:*** **satisfy a clause that is currently falsified**
- **Close branch if some clause is plainly falsified by it ($\star$)**

Propositional DPLL as a Semantic Tree Method

(1) $A \vee B$ (2) $C \vee \neg A$ (3) $D \vee \neg C \vee \neg A$ (4) $\neg D \vee \neg B$



$\{B\} \models A \vee B$

$\{B\} \models C \vee \neg A$

$\{B\} \models D \vee \neg C \vee \neg A$

$\{B\} \models \neg D \vee \neg B$

Model $\{B\}$ found.

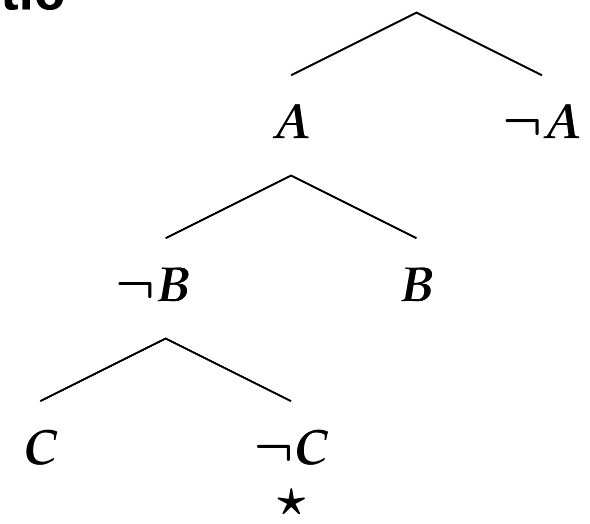
- **A Branch stands for an interpretation**
- ***Purpose of splitting:* satisfy a clause that is currently falsified**
- **Close branch if some clause is plainly falsified by it ($\star$)**

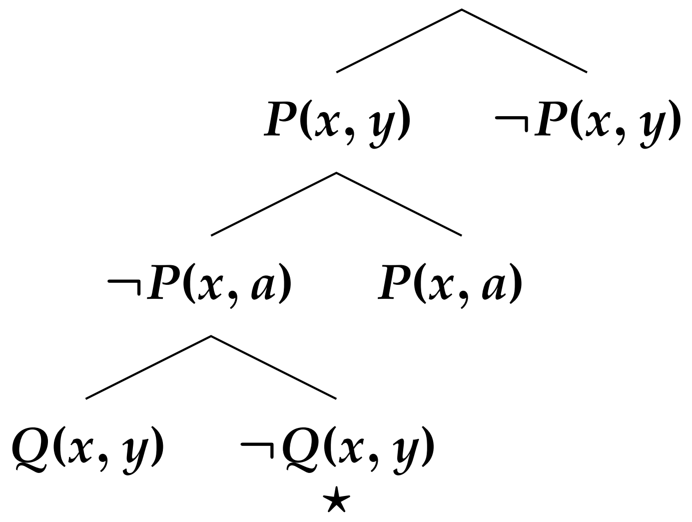
Meta-Level Strategy

Lifted data structures:

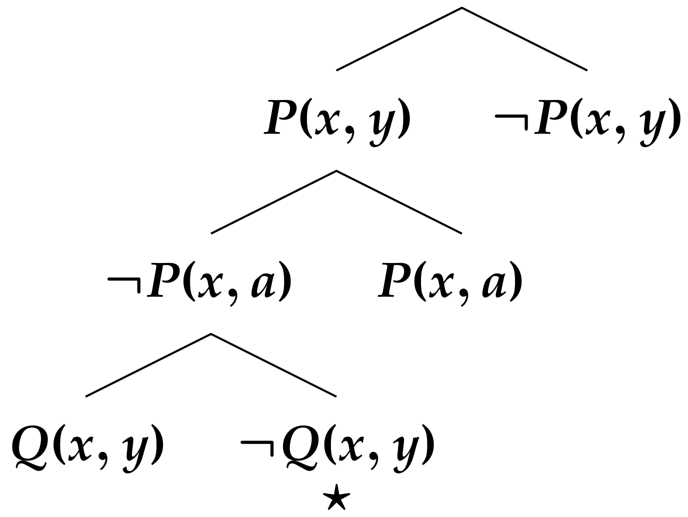
	DPLL	FDPLL
Clauses	$B \vee C$	$P(x, y) \vee Q(x, x)$

Semantic Trees





First-Order Semantic Trees



Issues:

- **How are variables treated?**

  (a) Universal?, (b) Rigid?, (c) Schematic!

- **What is the interpretation represented by a branch?**

  Clue to understanding of FDPLL (as is for Inst-Gen)

Extracting an Interpretation from a Branch

Branch B:

$P(x, y)$

Interpretation $I_B = \{\ldots\}$:

- **A branch literal specifies the truth values for all its ground instances, unless there is a more specific literal specifying the opposite truth value**

Extracting an Interpretation from a Branch

Branch B:

Interpretation $I_B = \{\ldots\}$:

$P(x, y)$ $\longrightarrow$

$P(a, a)$ $P(b, a)$

$P(a, b)$ $P(b, b)$

- **A branch literal specifies the truth values for all its ground instances, unless there is a more specific literal specifying the opposite truth value**

Extracting an Interpretation from a Branch

Branch B:

$P(x, y)$

$\neg P(a, y)$

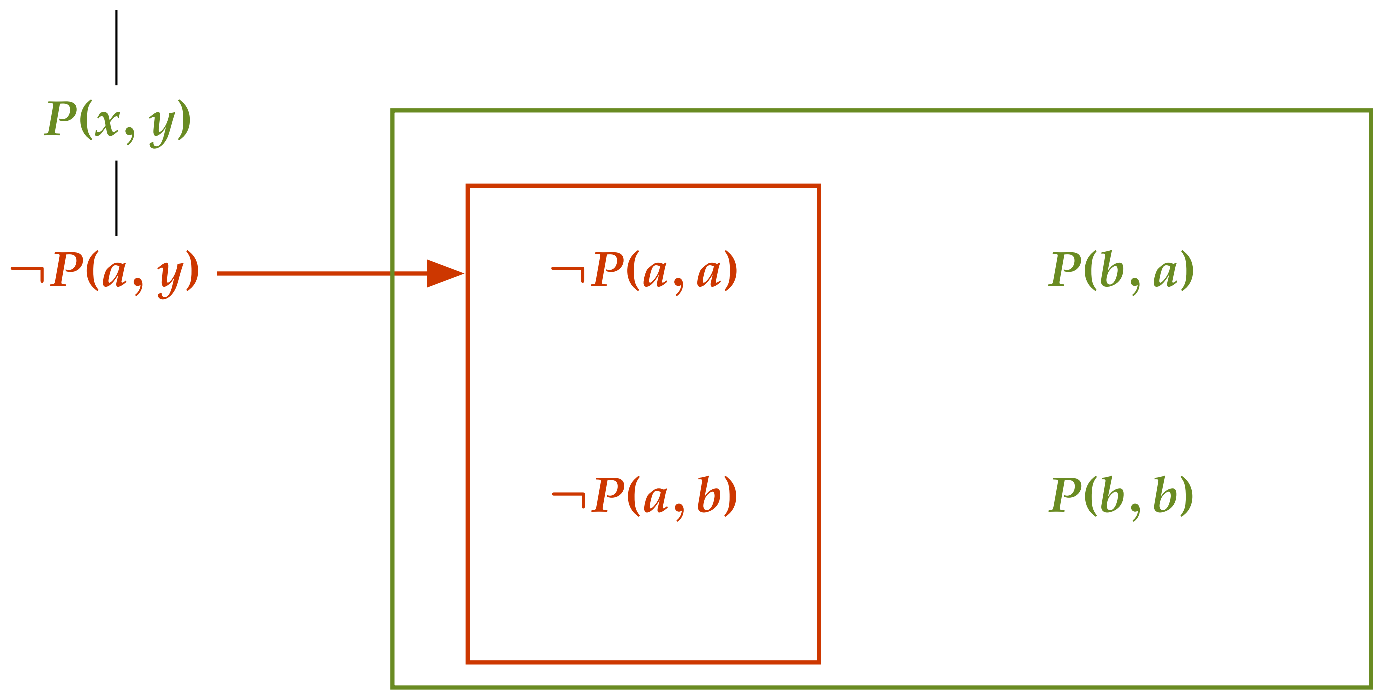
Interpretation $I_B = \{\ldots\}$:

$P(a, a)$	$P(b, a)$
$P(a, b)$	$P(b, b)$

- **A branch literal specifies the truth values for all its ground instances, unless there is a more specific literal specifying the opposite truth value**

Extracting an Interpretation from a Branch

Branch B: **Interpretation $I_B = \{\ldots\}$:**



- **A branch literal specifies the truth values for all its ground instances, unless there is a more specific literal specifying the opposite truth value**

Extracting an Interpretation from a Branch

Branch B:

$P(x, y)$

$\neg P(a, y)$

$\neg P(b, b)$

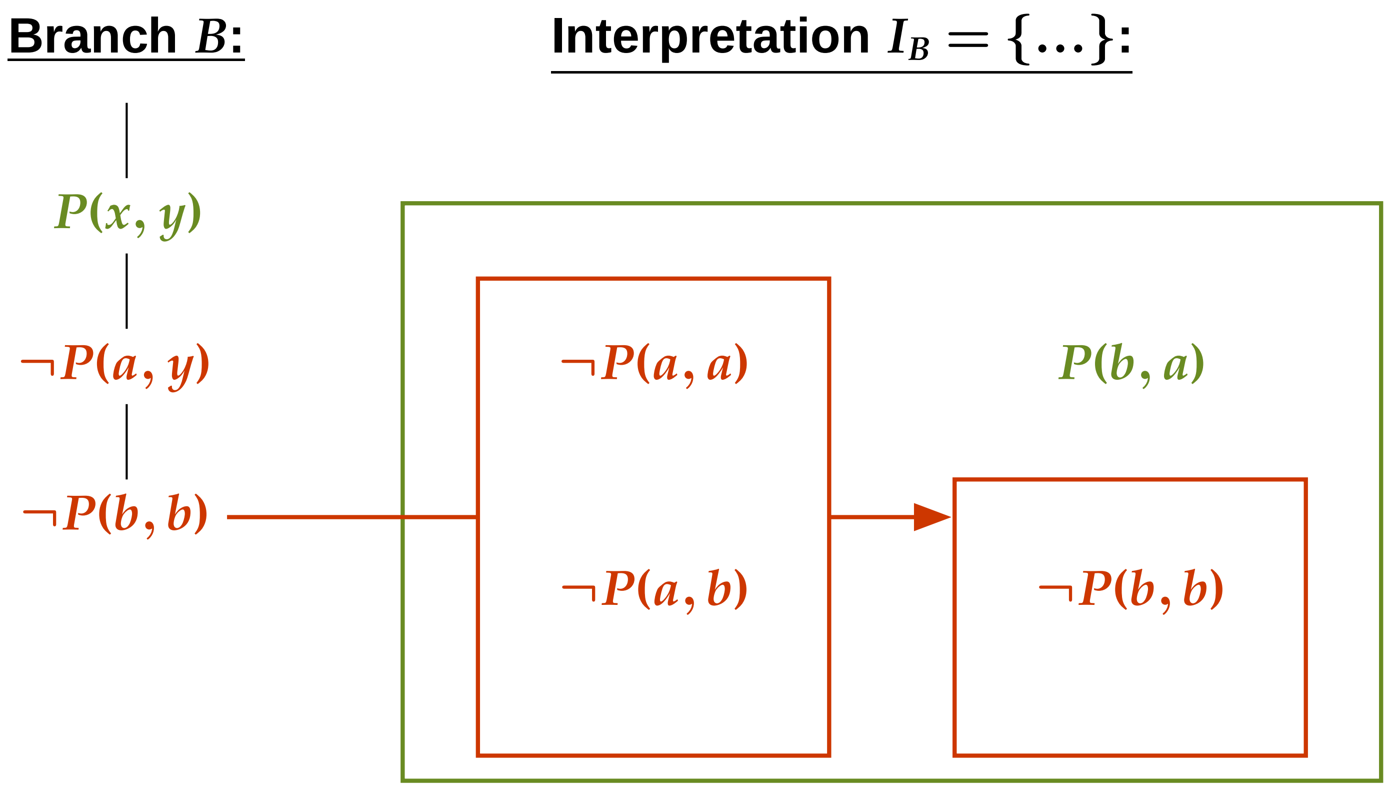
Interpretation $I_B = \{\ldots\}$:

$\neg P(a, a)$ $\quad$ $P(b, a)$

$\neg P(a, b)$ $\quad$ $P(b, b)$

- **A branch literal specifies the truth values for all its ground instances, unless there is a more specific literal specifying the opposite truth value**

Extracting an Interpretation from a Branch



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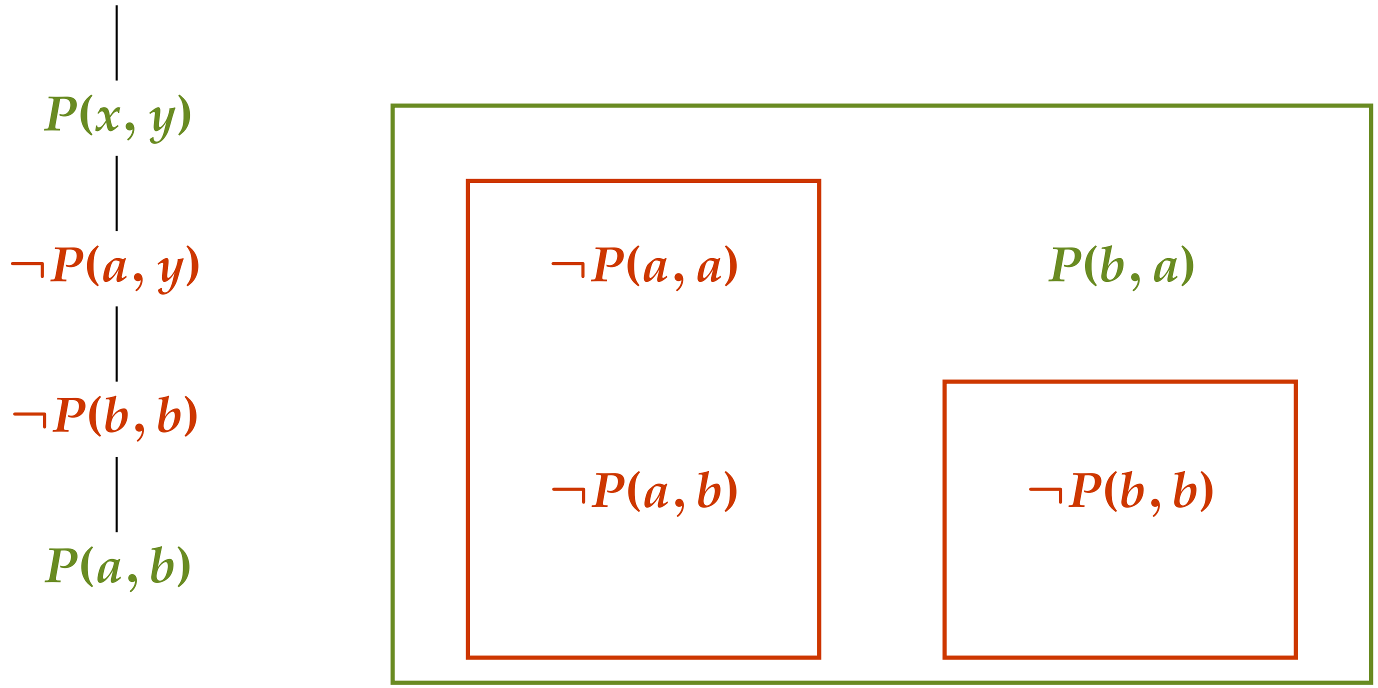

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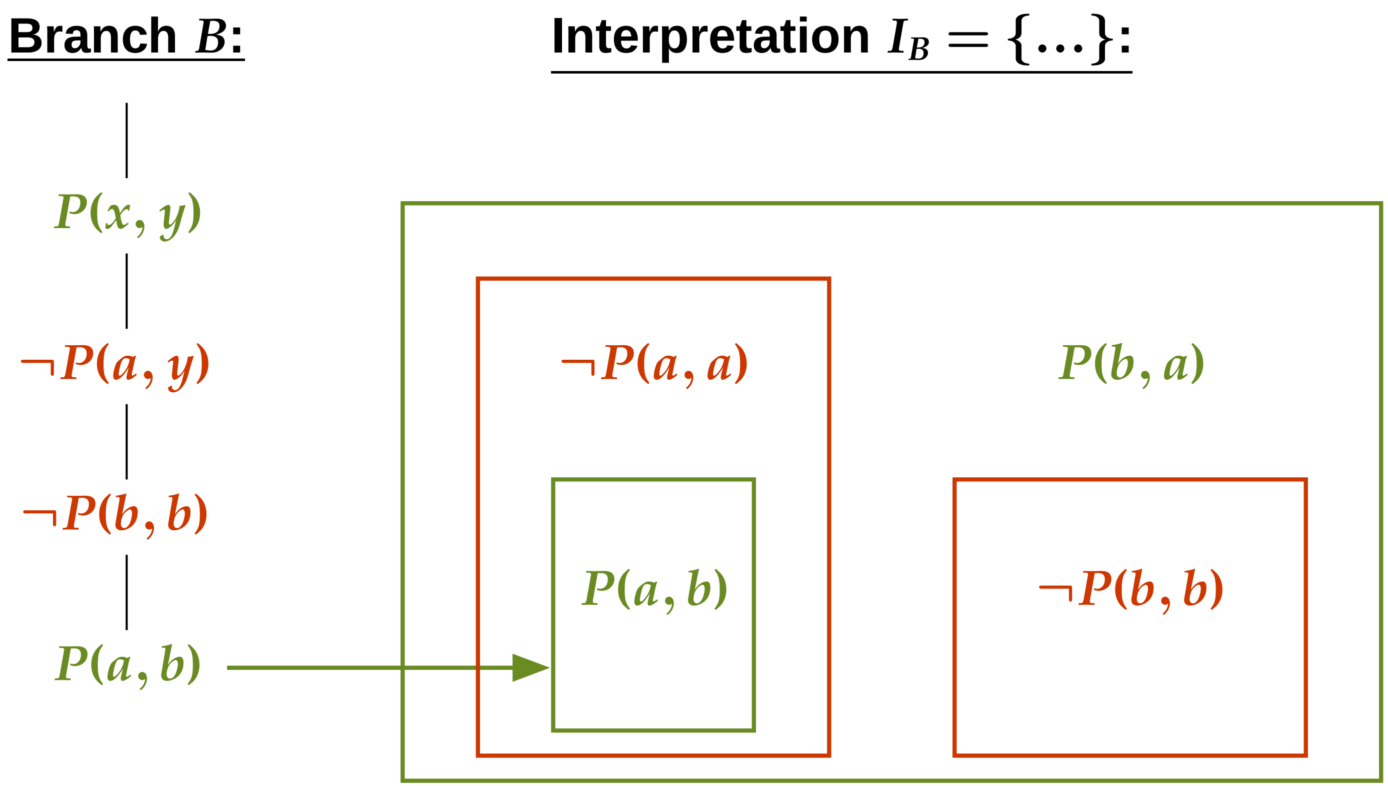


- **A branch literal specifies the truth values for all its ground instances, unless there is a more specific literal specifying the opposite truth value**

Extracting an Interpretation from a Branch

Branch B:

$$
\begin{array}{c}
| \\
P(x,y) \\
| \\
\neg P(a,y) \\
| \\
\neg P(b,b) \\
| \\
P(a,b)
\end{array}
$$

Interpretation $I_B = \{\ldots\}$:

$$
\{ \quad \neg P(a,a) \quad , \quad P(b,a) \quad , \quad P(a,b) \quad , \quad \neg P(b,b) \quad \}
$$

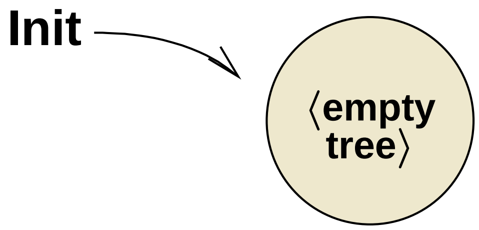
- A branch literal specifies the truth values for all its ground instances, unless there is a more specific literal specifying the opposite truth value
- The order of literals does not matter

FDPLL Calculus - Main Loop

Input: **a clause set S**

Output: **"unsatisfiable" or "satisfiable" (if it terminates)**

Note: Strategy much like in *inner* loop of propositional DPLL:



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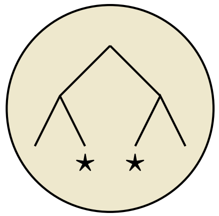

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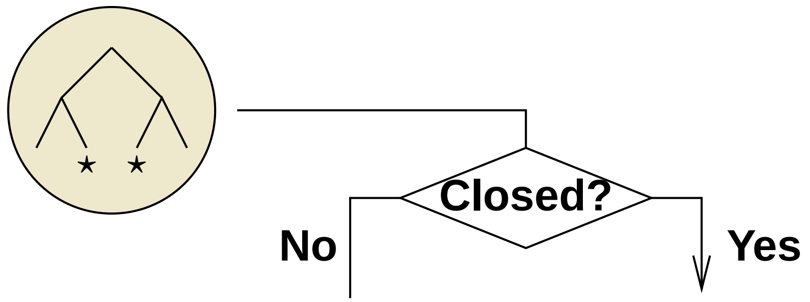

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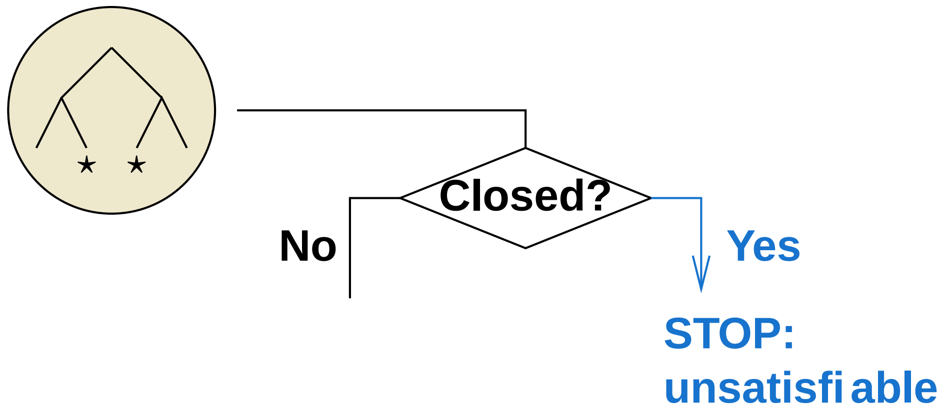

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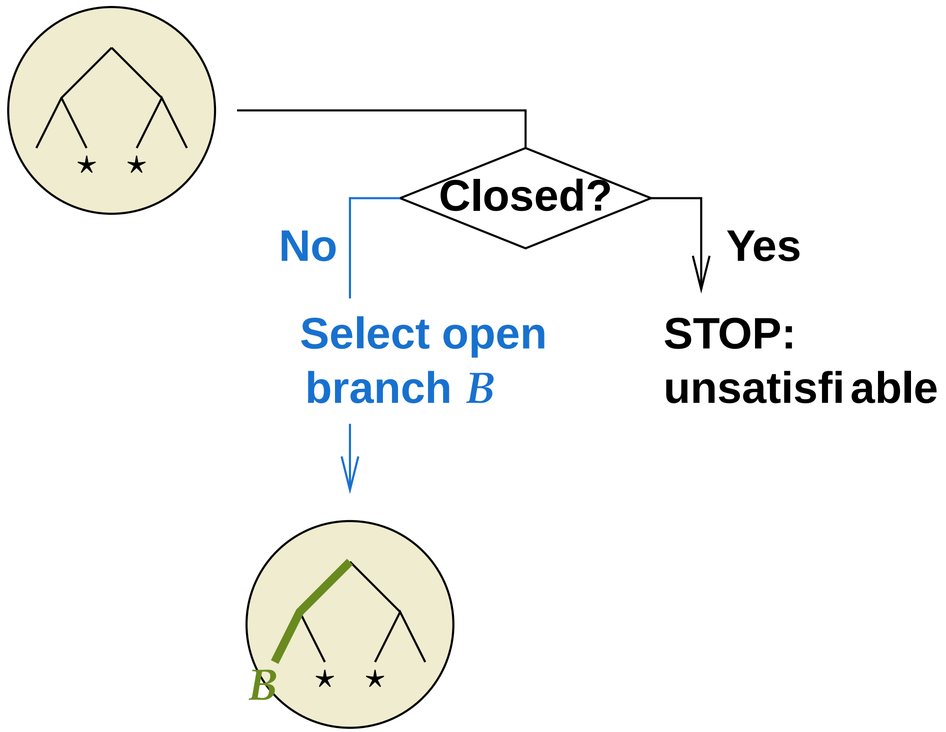

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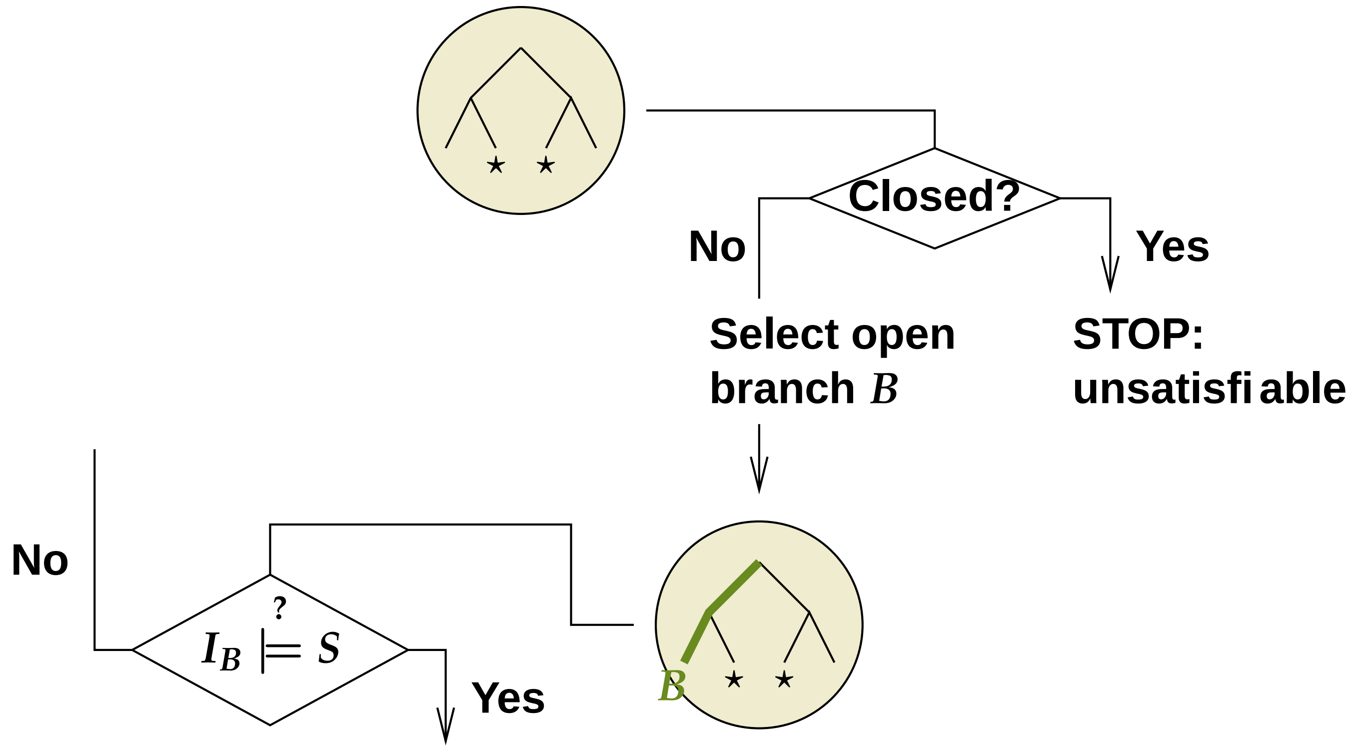

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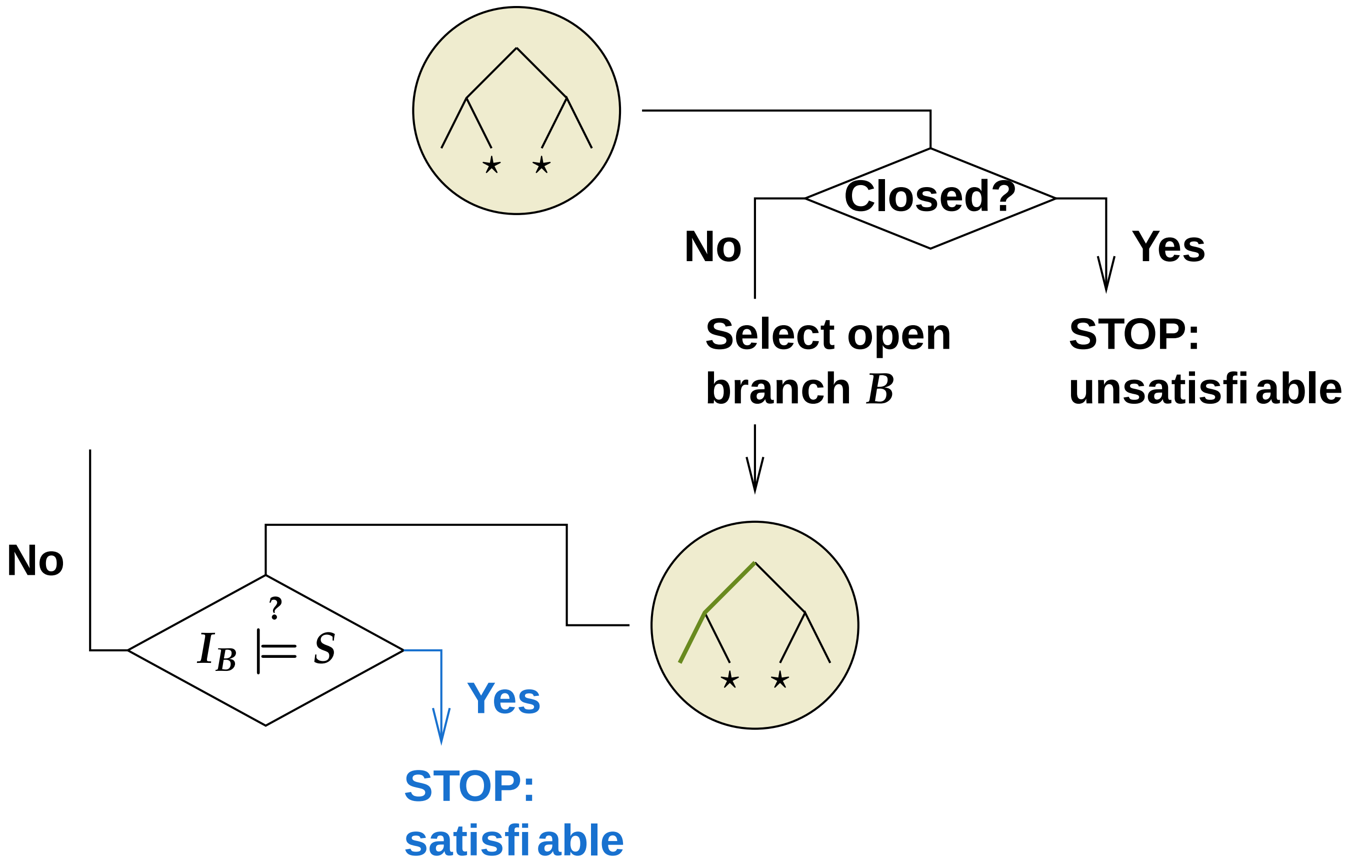

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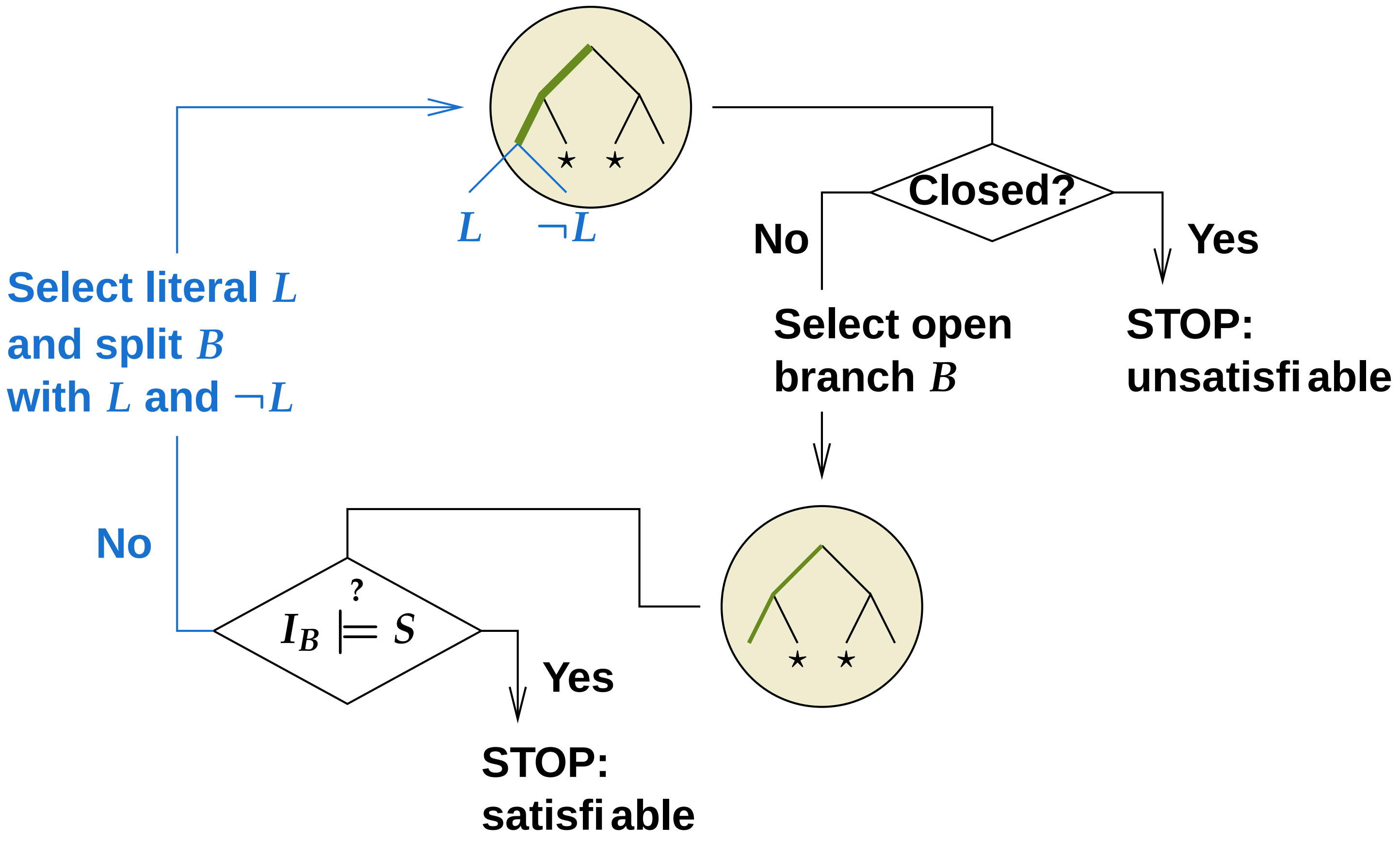

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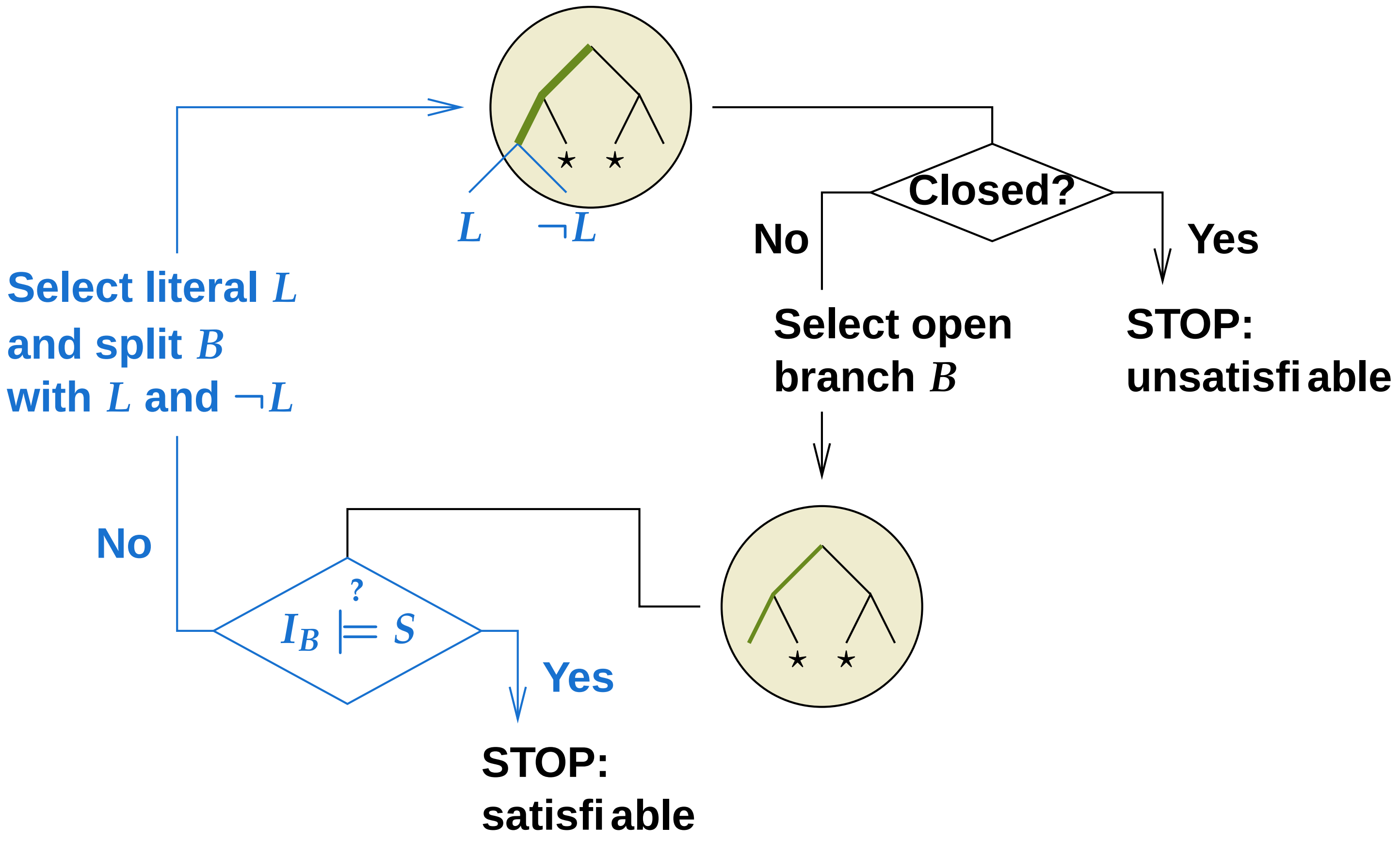

Not here: FDPLL derivation rules for testing $I_B \models S$ and Splitting

FDPLL – Model Computation Example

```
(1)  train(X,Y) ; flight(X,Y).       %% train from X to Y or flight from X to Y.
(2)  -flight(sb,X).                  %% no flight from sb to anywhere.
(3)  flight(X,Y) :- flight(Y,X).     %% flight is symmetric.
(4)  connect(X,Y) :- flight(X,Y).    %% a flight is a connection.
(5)  connect(X,Y) :- train(X,Y).     %% a train is a connection.
(6)  connect(X,Z) :- connect(X,Y),   %% connection is a transitive relation.
                     connect(Y,Z).
```

FDPLL – Model Computation Example

```
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                     connect(Y,Z).
```

Computed Model (as output by Darwin implementation)

```
+ flight(X, Y)
- flight(sb, X)
- flight(X, sb)
+ train(sb, Y)
+ train(Y, sb)
+ connect(X, Y)
```

FDPLL Model Computation Example - Derivation

•

Clause instance used in inference: $\boldsymbol{train(x,y) \lor flight(x,y)}$

FDPLL Model Computation Example - Derivation

$\mathit{flight}(x, y)$ $\qquad$ $\neg \mathit{flight}(x, y)$

Clause instance used in inference: $\neg \mathit{flight}(sb, x)$

FDPLL Model Computation Example - Derivation

$$\mathit{flight}(x, y) \qquad \neg \mathit{flight}(x, y)$$

$$\neg \mathit{flight}(sb, x) \qquad \mathit{flight}(sb, x)$$

Clause instance used in inference: $\mathit{train}(sb, y) \vee \mathit{flight}(sb, y)$

FDPLL Model Computation Example - Derivation

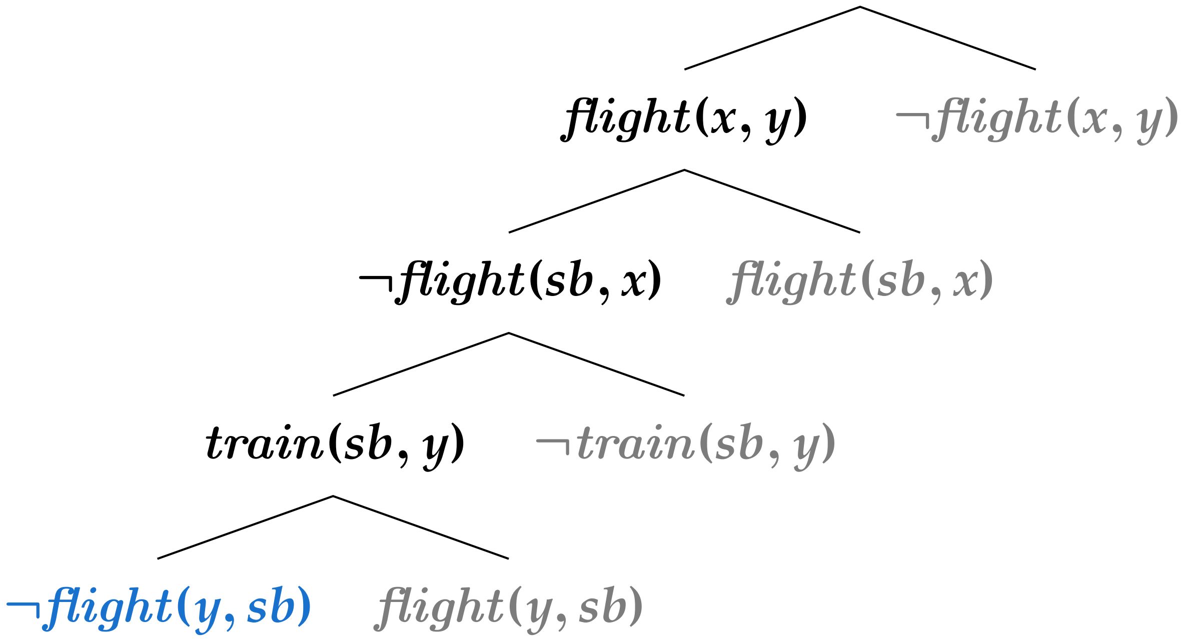
$$\mathit{flight}(x, y) \qquad \neg \mathit{flight}(x, y)$$

$$\neg \mathit{flight}(sb, x) \qquad \mathit{flight}(sb, x)$$

$$\mathit{train}(sb, y) \qquad \neg \mathit{train}(sb, y)$$

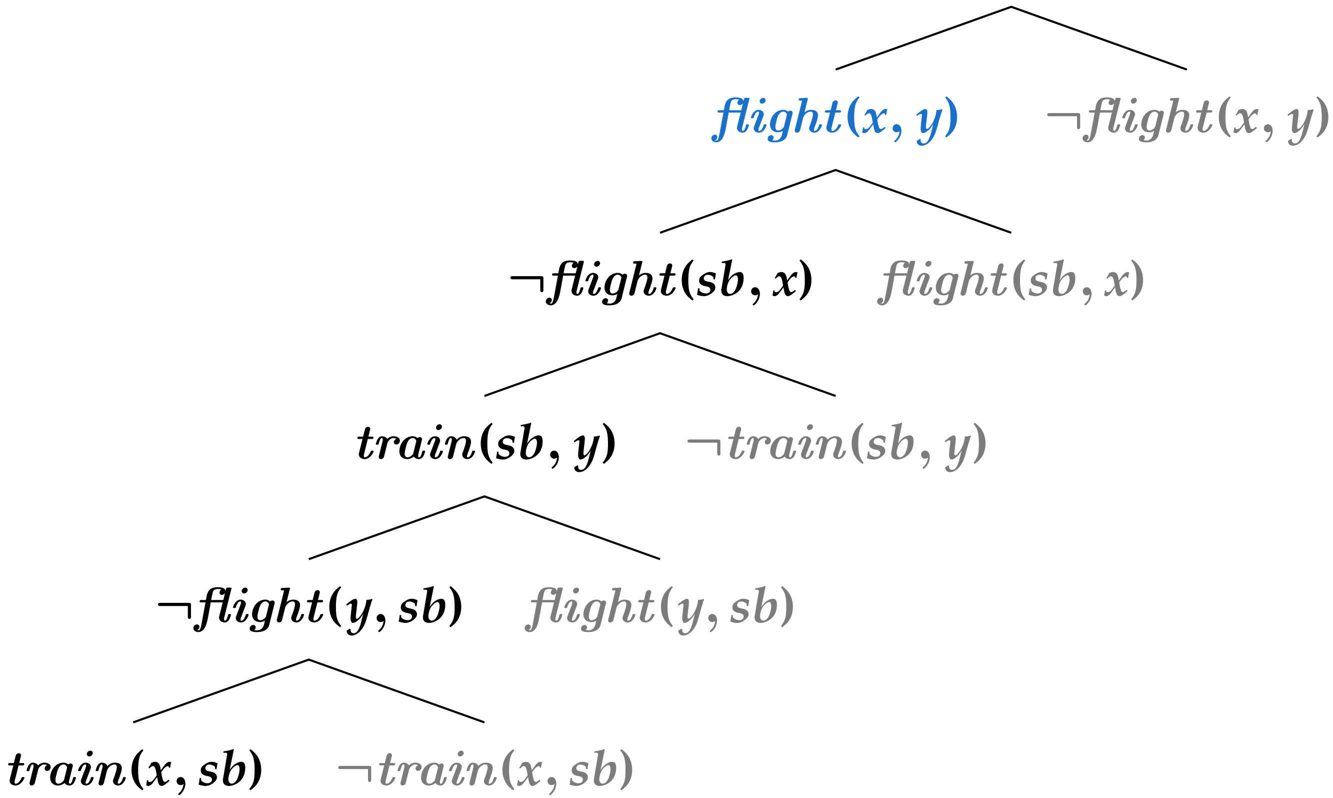
Clause instance used in inference: $\mathit{flight}(sb, y) \vee \neg \mathit{flight}(y, sb)$

FDPLL Model Computation Example - Derivation



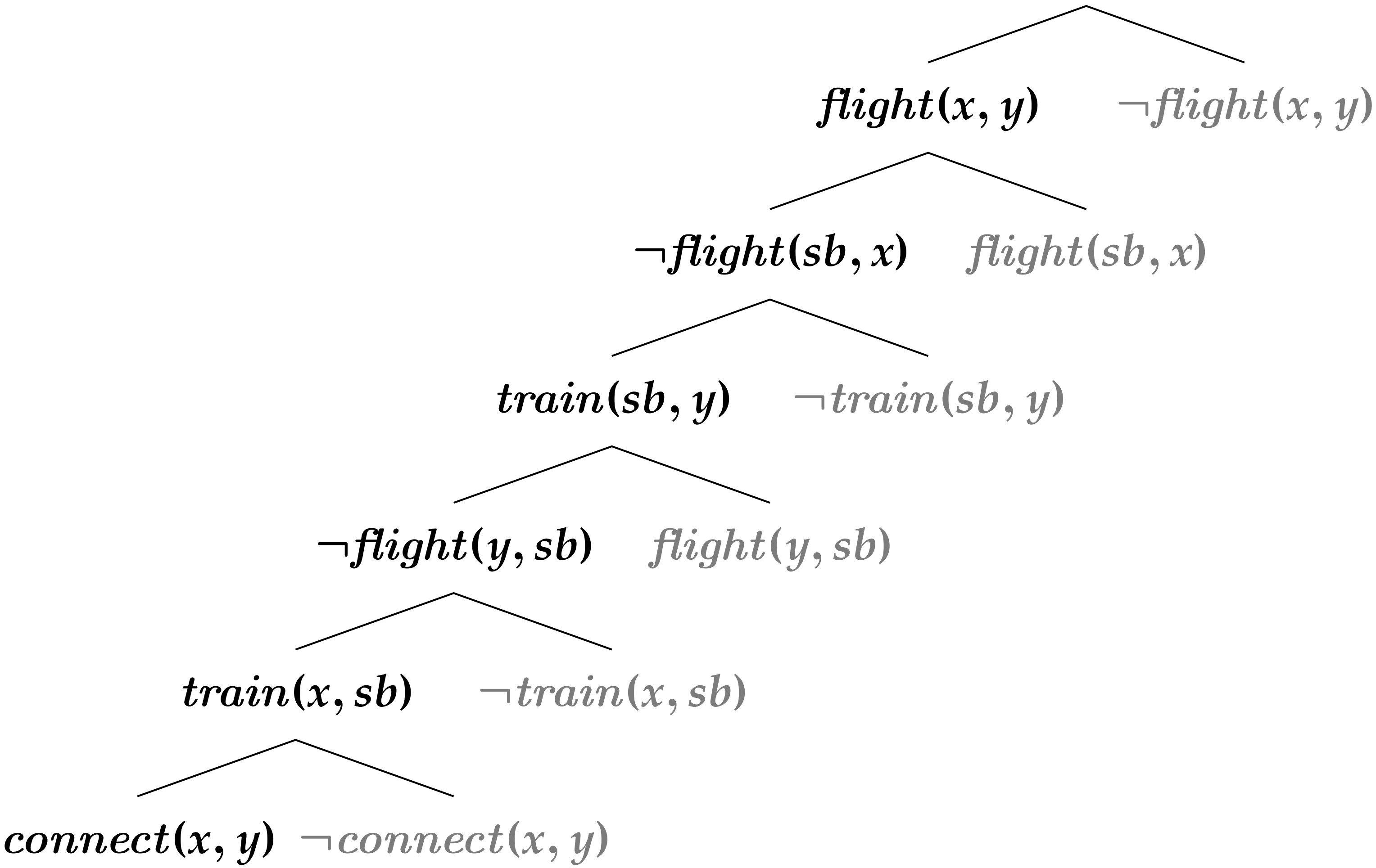
Clause instance used in inference: $train(x, sb) \vee flight(x, sb)$

FDPLL Model Computation Example - Derivation



Clause instance used in inference: $connect(x, y) \vee \neg flight(x, y)$

FDPLL Model Computation Example - Derivation



Done. **Return "satisfiable with model**

$\{flight(x, y), \ldots, connect(x, y)\}$"

Model Evolution (ME) Calculus

- **Same motivation as for FDPLL: lift propositional DPLL to first-order**
- **Loosely based on FDPLL, but wouldn't call it "extension"**
- **Extension of Tinelli's sequent-style DPLL [Tinelli, 2002]**
- **See [Baumgartner and Tinelli, 2003] for calculus, [Baumgartner *et al.*, 2005] for implementation "Darwin"**

Difference to FDPLL

- **Systematic treatment of universal and schematic variables**
- **Includes first-order versions of unit simplification rules**
- **Presentation as a sequent-style calculus, to cope with dynamically changing branches and clause sets due to simplification**

Model Evolution Calculus – Data Structure

- **Branches and clause sets may shrink as the derivation proceeds**
- **Such dynamics is best modeled with a sequent style calculus:**

$$\Lambda \vdash \Phi$$

Current Clause Set

Context: **A set of literals**
(the „current branch“)

Derivation Rules

- **Simplification rules**
- **Split**
- **Close**

Model Evolution Calculus – Data Structure

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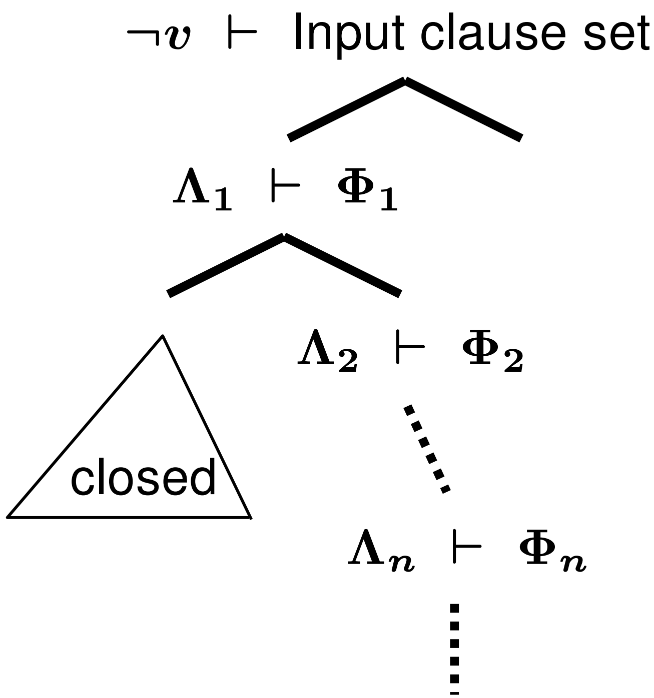
Derivations



Derivation Rules

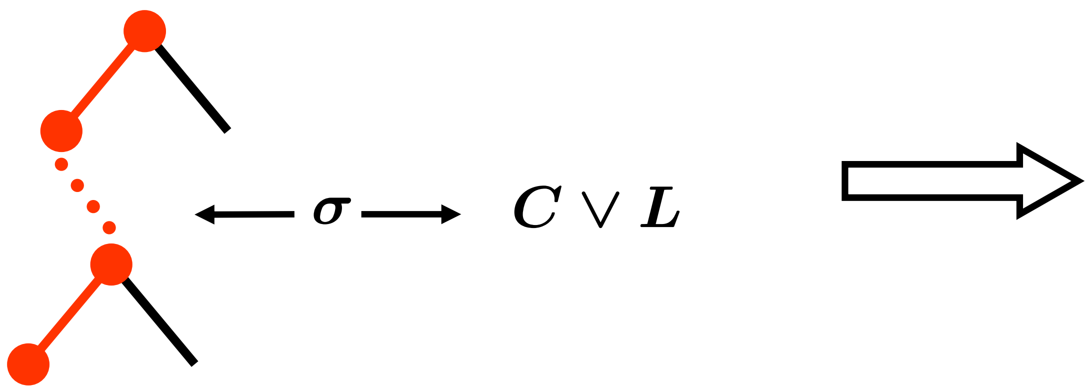
- **Simplification rules**
- **Split**
- **Close**

Derivation Rules - Split

$$\textbf{Split}\quad \frac{\Lambda \;\vdash\; \Phi,\, C \lor L}{\Lambda,\, L\sigma \;\vdash\; \Phi,\, C \lor L \qquad \Lambda,\, \overline{L}\sigma \;\vdash\; \Phi,\, C \lor L}$$

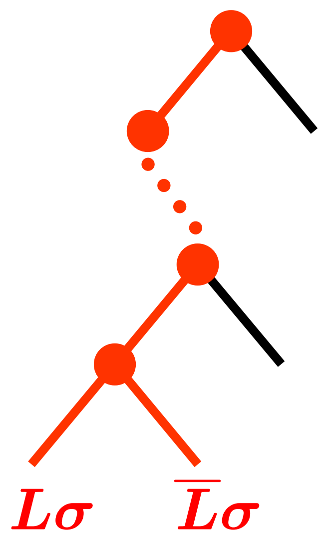
if

1. σ is a simultaneous mgu of $C \lor L$ against Λ,
2. neither $L\sigma$ nor $\overline{L}\sigma$ is contained in Λ, and
3. $L\sigma$ contains no variables (schematic variables OK, for simplicity here)



Λ current context

$\Phi, C \lor L$ current clause set



Derivation Rules – Split Example

$$\textbf{Split}\ \frac{\Lambda \ \vdash \ \Phi,\ C \vee L}{\Lambda,\ L\sigma \ \vdash \ \Phi,\ C \vee L \qquad \Lambda,\ \overline{L}\sigma \ \vdash \ \Phi,\ C \vee L}$$

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$$\Lambda: \quad P(u, u) \qquad Q(v, b)$$

$$C \vee L : \neg P(x, y) \vee \neg Q(a, z)$$

Derivation Rules – Split Example

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$$\Lambda: \quad P(u,u) \qquad Q(v,b)$$

$$C \vee L : \neg P(x,y) \vee \neg Q(a,z)$$

$$\sigma = \{\, x \mapsto u,\ y \mapsto u,\ v \mapsto a,\ z \mapsto b \,\}$$

$$(C \vee L)\sigma : \neg P(x,x) \vee \neg Q(a,b)$$

Derivation Rules – Split Example

$$\textbf{Split}\ \frac{\Lambda \ \vdash\ \Phi,\ C \vee L}{\Lambda,\ L\sigma\ \vdash\ \Phi,\ C \vee L \qquad \Lambda,\ \overline{L}\sigma\ \vdash\ \Phi,\ C \vee L}$$

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$$\Lambda\text{:} \quad \underline{P(u,u)} \qquad \underline{Q(v,b)}$$

$$C \vee L : \neg P(x,y) \vee \neg Q(a,z)$$

$$\sigma = \{\, x \mapsto u,\ y \mapsto u,\ v \mapsto a,\ z \mapsto b \,\}$$

$$(C \vee L)\sigma : \neg \underline{P(x,x)} \vee \neg \underline{Q(a,b)}$$

2. violated **2. satisfied**

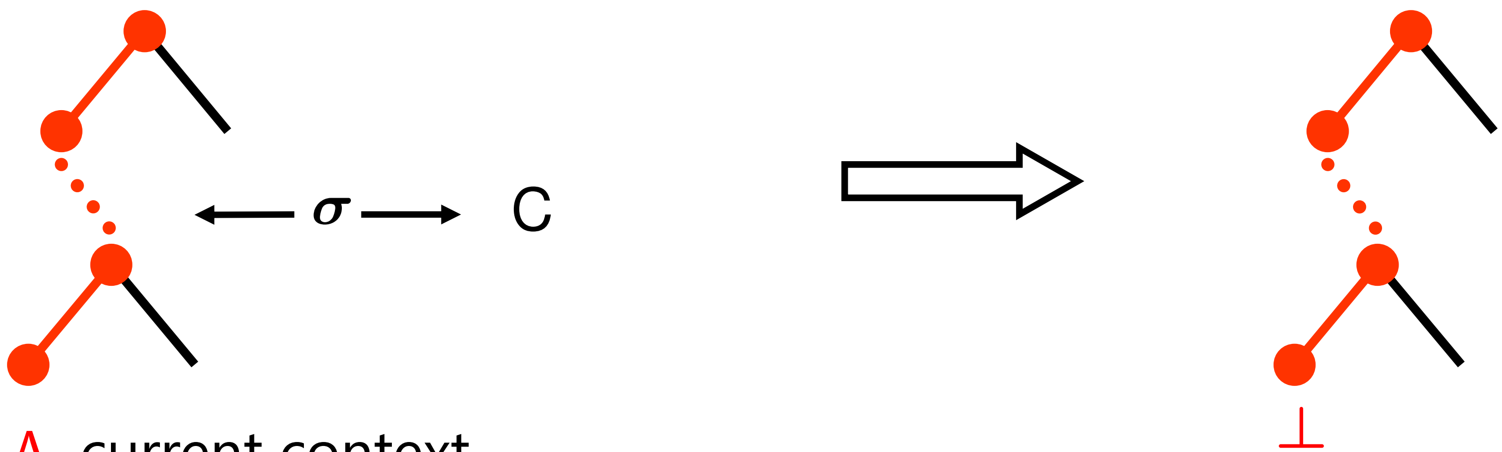
$L\sigma = \neg Q(a,b)$ is admissible for Split

Derivation Rules – Close

$$\textbf{Close}\ \frac{\Lambda \ \vdash \ \Phi,\ C}{\Lambda \ \vdash \ \bot}$$

if

1. $\Phi \neq \emptyset$ or $C \neq \bot$, and
2. there is a simultaneous mgu σ of C against Λ such that Λ contains the complement of each literal of $C\sigma$



Λ current context

Φ, C current clause set

Derivation Rules – Close Example

Close $\dfrac{\Lambda \;\vdash\; \Phi,\, C}{\Lambda \;\vdash\; \bot}$

if

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Λ: $P(u, u) \quad Q(a, b)$

$C : \neg P(x, y) \vee \neg Q(a, z)$

Derivation Rules – Close Example

$$\textbf{Close}\ \frac{\Lambda \quad \vdash \quad \Phi,\ C}{\Lambda \quad \vdash \quad \bot}$$

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Λ: $\quad P(u,u) \qquad Q(a,b)$

$$\sigma = \{\, x \mapsto u,\ y \mapsto u,\ z \mapsto b \,\}$$

$C : \neg P(x,y) \vee \neg Q(a,z)$

$C\sigma : \neg P(x,x) \vee \neg Q(a,b)$

Derivation Rules – Close Example

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$$\Lambda: \quad \underline{P(u,u)} \qquad \underline{Q(a,b)}$$

$$C : \neg P(x,y) \vee \neg Q(a,z)$$

$$\sigma = \{\, x \mapsto u,\ y \mapsto u,\ z \mapsto b \,\}$$

$$C\sigma : \neg \underline{P(x,x)} \vee \neg \underline{Q(a,b)}$$

2. satisfied **2. satisfied**

Close is applicable

Derivation Rules – Simplification Rules (1)

Propositional level:

$$\textbf{Subsume} \quad \frac{\Lambda,\ L \quad \vdash \quad \Phi,\ L \vee C}{\Lambda,\ L \quad \vdash \quad \Phi}$$

First-order level $\approx$ unit subsumption:

- All variables in context literal L must be universally quantified
- Replace equality by matching

Derivation Rules – Simplification Rules (2)

Propositional level:

Resolve
$$\frac{\Lambda,\ L \quad \vdash \quad \Phi,\ \overline{L} \lor C}{\Lambda,\ L \quad \vdash \quad \Phi,\ C}$$

First-order level $\approx$ restricted unit resolution

- All variables in context literal L must be universally quantified
- Replace equality by unification
- The unifier must not modify C

Derivation Rules – Simplification Rules (3)

Compact $$\frac{\Lambda,\ K,\ L \quad \vdash \quad \Phi}{\Lambda,\ K \qquad \vdash \quad \Phi}$$

if

1. all variables in K are universally quantified
2. $K\sigma = L$, for some substitution σ

Derivations and Completeness

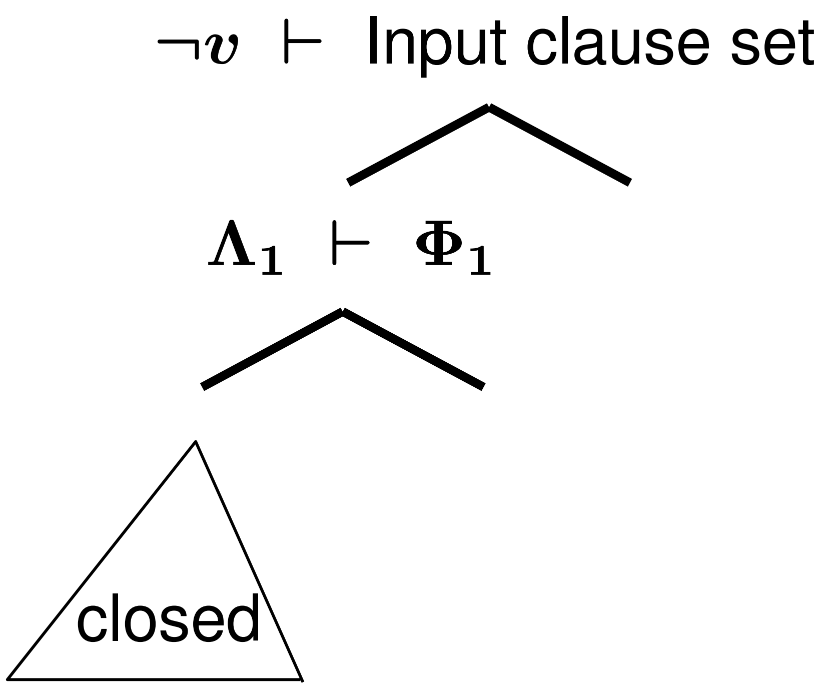
$\neg v \;\vdash\;$ Input clause set

Derivations and Completeness

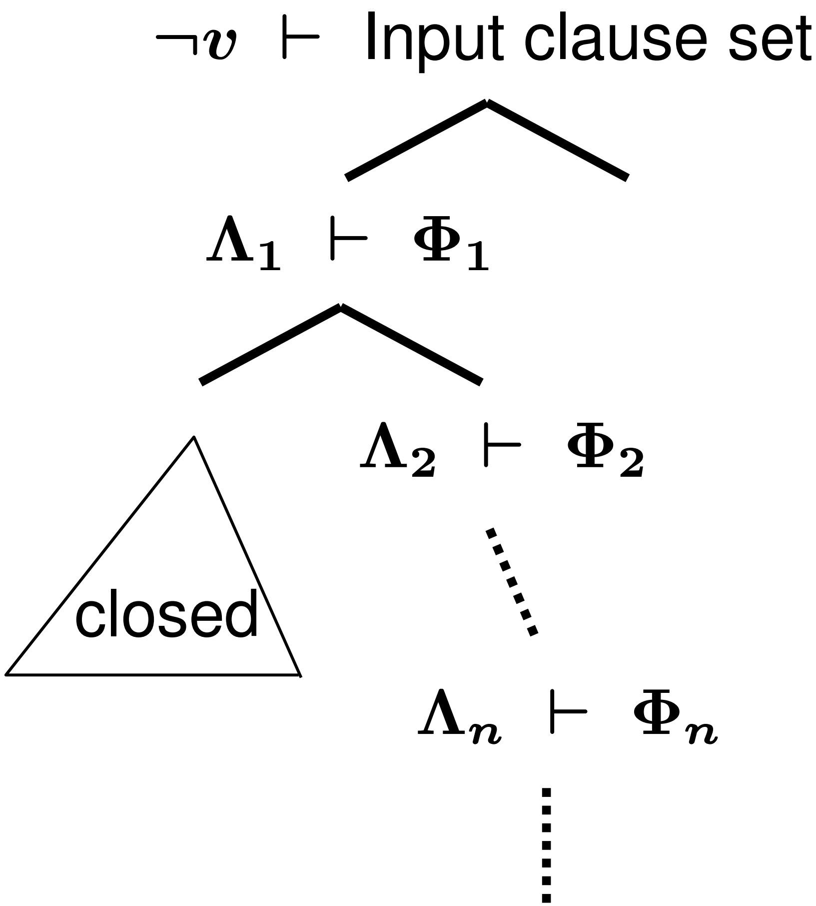
$\neg v \ \vdash$ Input clause set

$\Lambda_1 \ \vdash \ \Phi_1$

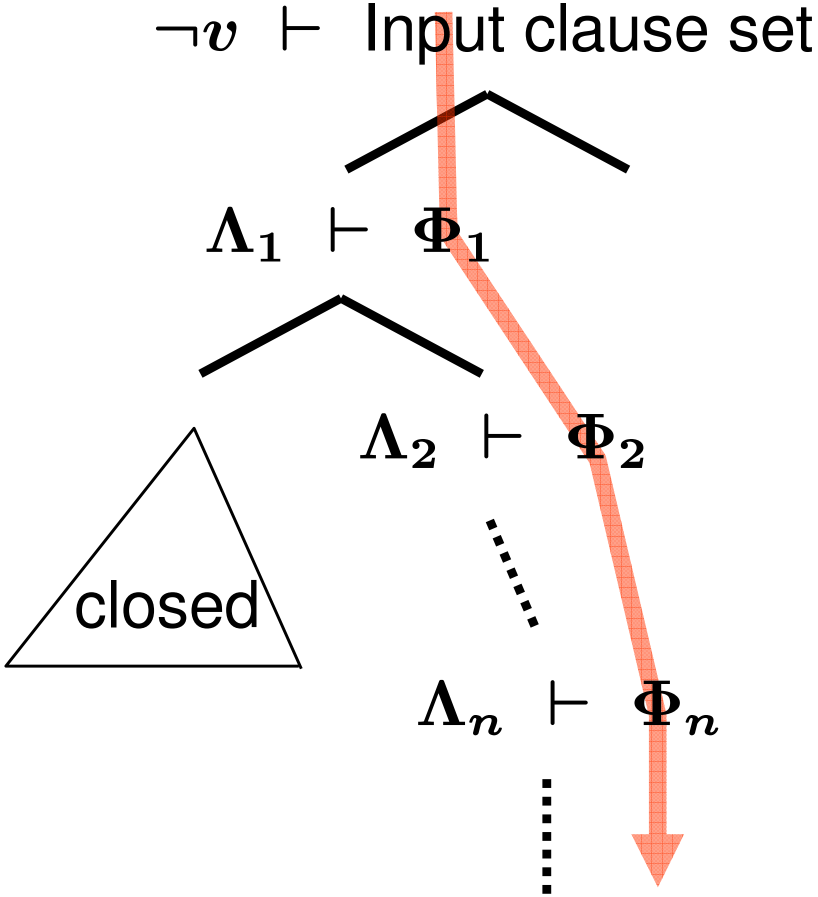
Derivations and Completeness



Derivations and Completeness

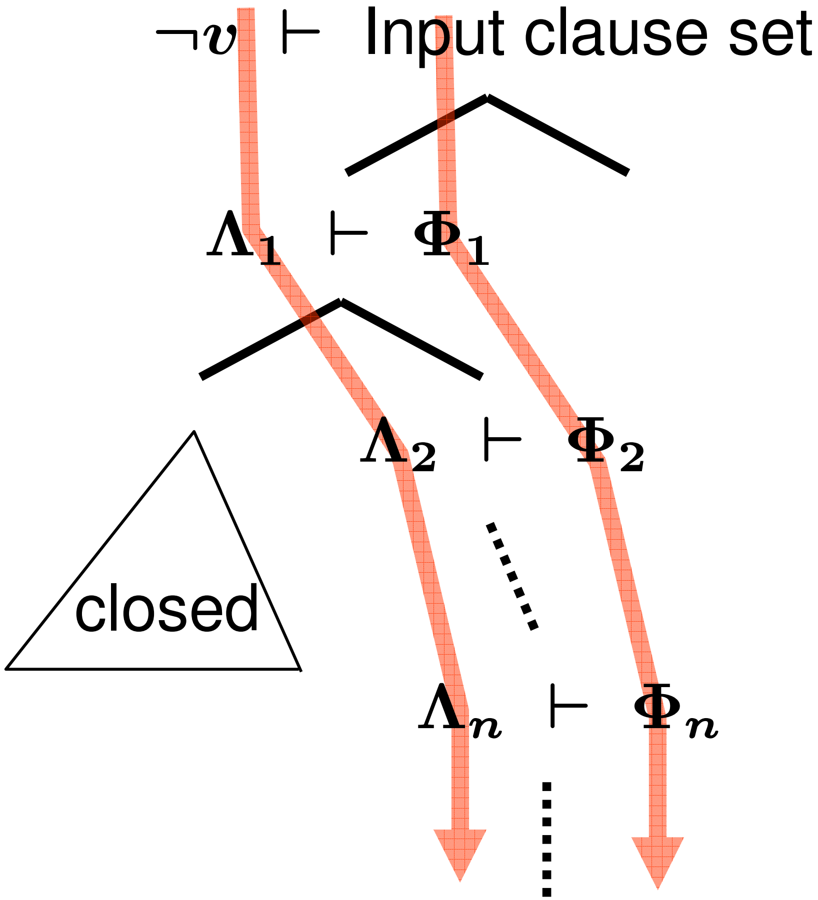


Derivations and Completeness



$$\Phi_\infty := \bigcup_{i \geq 0} \bigcap_{j \geq i} \Phi_j$$

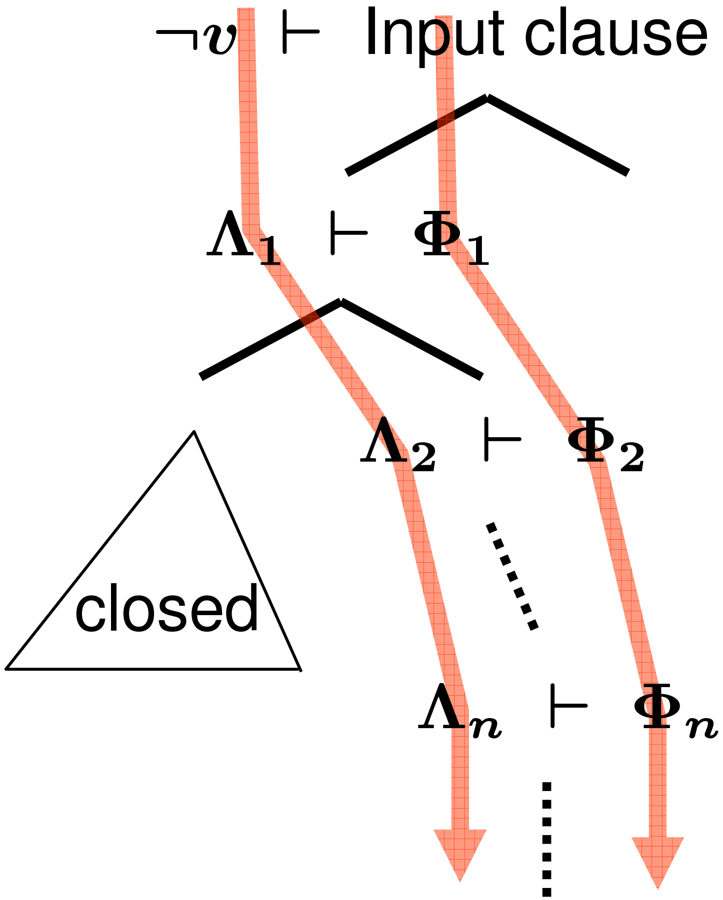
Derivations and Completeness



$$\Lambda_\infty := \bigcup_{i\geq 0} \bigcap_{j\geq i} \Lambda_j$$

$$\Phi_\infty := \bigcup_{i\geq 0} \bigcap_{j\geq i} \Phi_j$$

Derivations and Completeness



$$\Lambda_\infty := \bigcup_{i \geq 0} \bigcap_{j \geq i} \Lambda_j$$

$$\Phi_\infty := \bigcup_{i \geq 0} \bigcap_{j \geq i} \Phi_j$$

Fairness

Closed tree or open limit tree,
with some branch satisfying:

1. Close not applicable to any Λ_i
2. For all $C \in \Phi_\infty$ and subst. γ,
   "if for some i, $\Lambda_i \not\models C\gamma$
   then there is $j \geq i$
   such that $\Lambda_j \models C\gamma$

(Use Split to achieve this)

Completeness

Suppose a fair derivation
of an open limit tree

Show that $\Lambda_\infty \models \Phi_\infty$

Implementation: Darwin

- „Serious“ Implementation

  Part of Master Thesis, continued in Ph.D. project (A. Fuchs)
- (Intended) Applications
  - detecting dependent variables in CSP problems
  - strong equivalence of logic programs
  - Finite countermodels for program verification purposes
  - Bernays-Schoenfinkel fragment of autoepistemic logic
- Currently extended:
  - Lemma learning
  - Equality inference rules [Baumgartner and Tinelli, 2005]
- Written in OCaml, 14K LOC
- User manual, proof tree output (GraphViz)
- Download at `http://goedel.cs.uiowa.edu/Darwin/`

FDPLL/ME vs. OSHL

Recall OSHL:

- Stepwisely modify I_0
  Modified interpretation represented as $I_0(L_1, \ldots, L_m)$
- Find next ground instance $C\gamma$ by unifying subclause of C against $(L_1, \ldots, L_m)$ and guess Herbrand-instantiation of rest clause, so that $I_0(L_1, \ldots, L_m) \not\models C\gamma$

FDPLL/ME

- Initial interpretation I_0 is a trival one (e.g. “false everywhere”)
- But $(L_1, \ldots, L_m)$ is a set of first-order literals now
- Find next (possibly) non-ground instance $C\sigma$ by unifying C against $(L_1, \ldots, L_m)$ so that $(L_1, \ldots, L_m) \not\models C\sigma$

FDPLL/ME vs. Inst-Gen

FDPLL/ME and Inst-Gen temporarily switch to propositional reasoning.

But:

Inst-Gen (and other two-level calculi)

- **Use the $\perp$-version $S_\perp$ of the current clause set S**

$\Rightarrow$ **Works globally, on clause sets**

- **Flexible: may switch focus all the time – but memory problem (?)**

FDPLL/ME (and other one-level calculi)

- **Use the $\$$-version of the current branch**

$\Rightarrow$ **Works locally in context of current branch**

- **Not so flexible – but don't expect memory problems:**
  FDPLL/ME need not keep *any* clause instance
  DCTP needs to keep clause instances only along current branch

Applicability/Non-Applicability of IMs

- **Comparison: Resolution vs. Tableaux vs. IMs**
- **Conclusions from that**

Resolution vs. Tableaux vs. IMs

Consider a transitivity clause $P(x,z) \leftarrow P(x,y) \land P(y,z)$

Resolution

- **Resolution may generate clauses of unbounded length:**

$$P(x,z') \leftarrow P(x,y) \land P(y,z) \land P(z,z')$$

$$P(x,z'') \leftarrow P(x,y) \land P(y,z) \land P(z,z') \land P(z',z'')$$

- Does not decide function-free clause sets

- Complicated to extract model

+ (Ordered) Resolution very good on some classes, Equality

Resolution vs. Tableaux vs. IMs

Consider a transitivity clause $P(x, z) \leftarrow P(x, y) \wedge P(y, z)$

Rigid Variables Approaches (Tableaux, Connection Methods)

- Have to use unbounded number of variants per clause:

$$
\begin{aligned}
P(x', z') &\leftarrow P(x', y') \wedge P(y', z') \\
P(x'', z'') &\leftarrow P(x'', y'') \wedge P(y'', z'')
\end{aligned}
$$

  - Weak redundancy criteria
  - Difficult to exploit proof confluence

  Usual calculi backtrack more than theoretically necessary

  But see [Giese, 2001], [Baumgartner *et al.*, 1999], [Beckert, 2003]

- Model Elimination: *goal-orientedness* compensates drawback

Difficulty with Rigid Variable Methods

Rigid variable methods "destructively" modify data structure

S: $\forall x (P(x) \vee Q(x))$
$\neg P(a)$
$\neg P(b)$
$\neg Q(b)$

(1) $P(X) \vee Q(X)$

(2) $P(X) \vee Q(X)$
$\neg P(a)$

(3) $P(a) \vee Q(a)$
$\neg P(a)$

(5) $P(a) \vee Q(a)$
$\neg P(a)$
$P(X') \vee Q(X')$
$\neg P(b)$

(7) $P(a) \vee Q(a)$
$\neg P(a)$
$P(b) \vee Q(b)$
$\neg P(b)$
$\neg Q(b)$

- **Connection method (and tableaux) are proof confluent: no deadends**
- **Difficulty to find fairness criterion due to "destructive" nature**
- **All IMs are non-destructive – no problem here**

Resolution vs. Tableaux vs. IMs

Consider a transitivity clause $P(x,z) \leftarrow P(x,y) \wedge P(y,z)$

Instance Based Methods

- May need to generate and keep *proper* instances of clauses:

$$P(x,z) \leftarrow P(x,y) \wedge P(y,z)$$

$$P(a,z) \leftarrow P(a,y) \wedge P(y,b)$$

- \- Cannot use subsumption: weaker than Resolution
- \- Clauses do not grow in length, no recombination of clauses: better than Resolution, same as in rigid variables approaches
- \+ Need not keep variants: better than rigid variables approaches

Applicability/Non-Applicability of IMs: Conclusions

Suggested applicability for IMs:

- Near propositional clause sets
- Clause sets without function symbols (except constants)

  E.g. Translation from basic modal logics, Datalog
- Model computation (sometimes)

Other methods (currently?) better at:

- Goal orientation
- Equality, theory reasoning
- Many decidable fragments (Guarded fragment, two-variable fragment)

Open Research Problem

- **ARM (atomic representation of models) [Gottlob and Pichler, 1998]**
  ARM: set of atoms. Set of all ground instances is an interpretation
- **Contexts are stronger than ARMs. E.g., for $\Lambda = \{P(u,v), \neg P(u,u)\}$ and $\Sigma_F = \{a/0, f/1\}$ there is no equivalent ARM**
- **Contexts are equivalent to DIGs (Disjunctions of Implicit Generalizations) [Fermüller and Pichler, 2005]**
- **Contexts cannot represent certain infinite interpretations, e.g. minimal models of the clause set**

$$P(x) \vee P(f(x)),\ \neg P(x) \vee \neg P(f(x))$$

Open Research Problem

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$$P(x) \vee P(f(x)),\ \neg P(x) \vee \neg P(f(x))$$

Instance Based Method based on more powerful model representation?

Part II: A Closer Look

- Disconnection calculus
- Theory Reasoning and Equality
- Implementations and Techniques
  - Available Implementations
  - Proof Procedures
  - Exploiting SAT techniques

Disconnection Tableaux

The Disconnection Calculus(I)

- Analytic tableau calculus for first order clause logic
- Introduced by J.-P. Billon (1996)
- Special characteristics of calculus:
  - No *rigid* variables
  - No *variants* in tableau
  - *Proof confluence*: One proof tree only, no backtracking in search
  - *Saturated* branches as indicator of satisfiability
  - *Decision procedure* for certain classes of formulae
- Related methods: hyper linking, hyper tableaux, first order Davis-Putnam . . .

The Disconnection Calculus (II)

- **Singular inference rule: Linking**

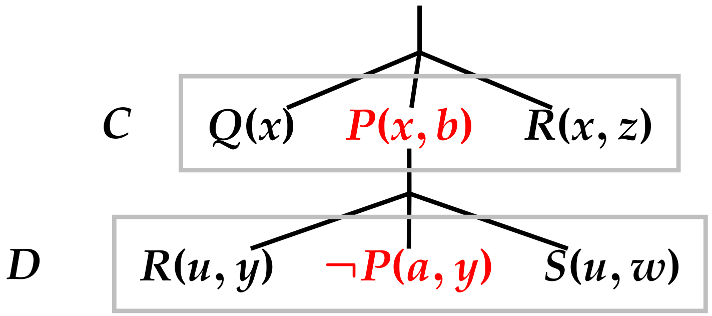
C $\boxed{Q(x) \quad P(x,b) \quad R(x,z)}$

D $\boxed{R(u,y) \quad \neg P(a,y) \quad S(u,w)}$

potentially complementary literals on path

The Disconnection Calculus (II)

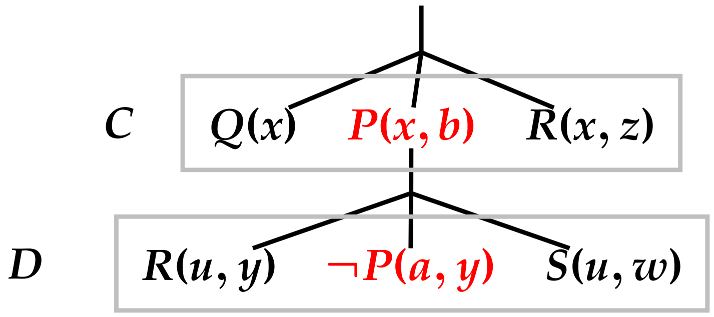
- **Singular inference rule: Linking**

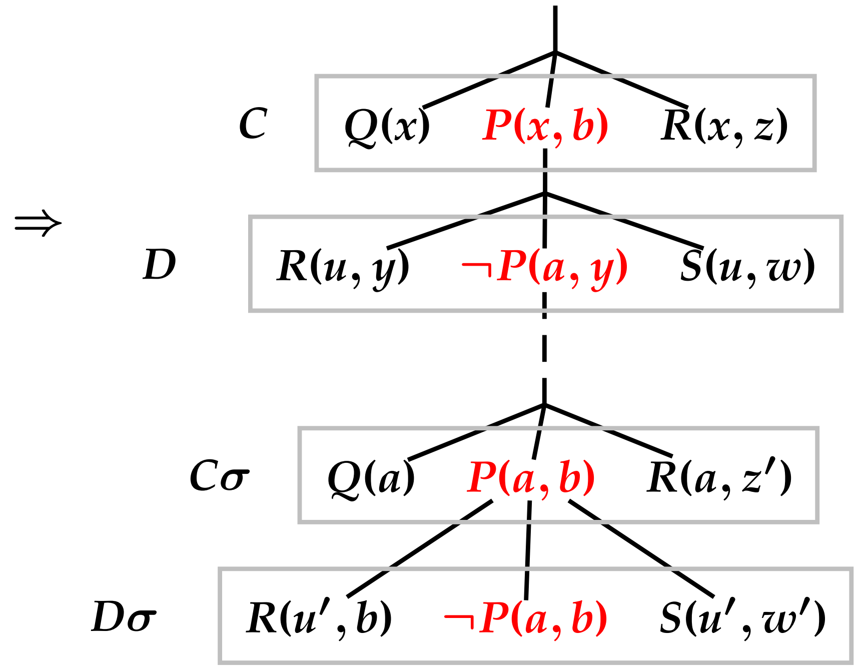


unifier for literals:
$\{x/a, y/b\}$

The Disconnection Calculus (II)

- **Singular inference rule: Linking**

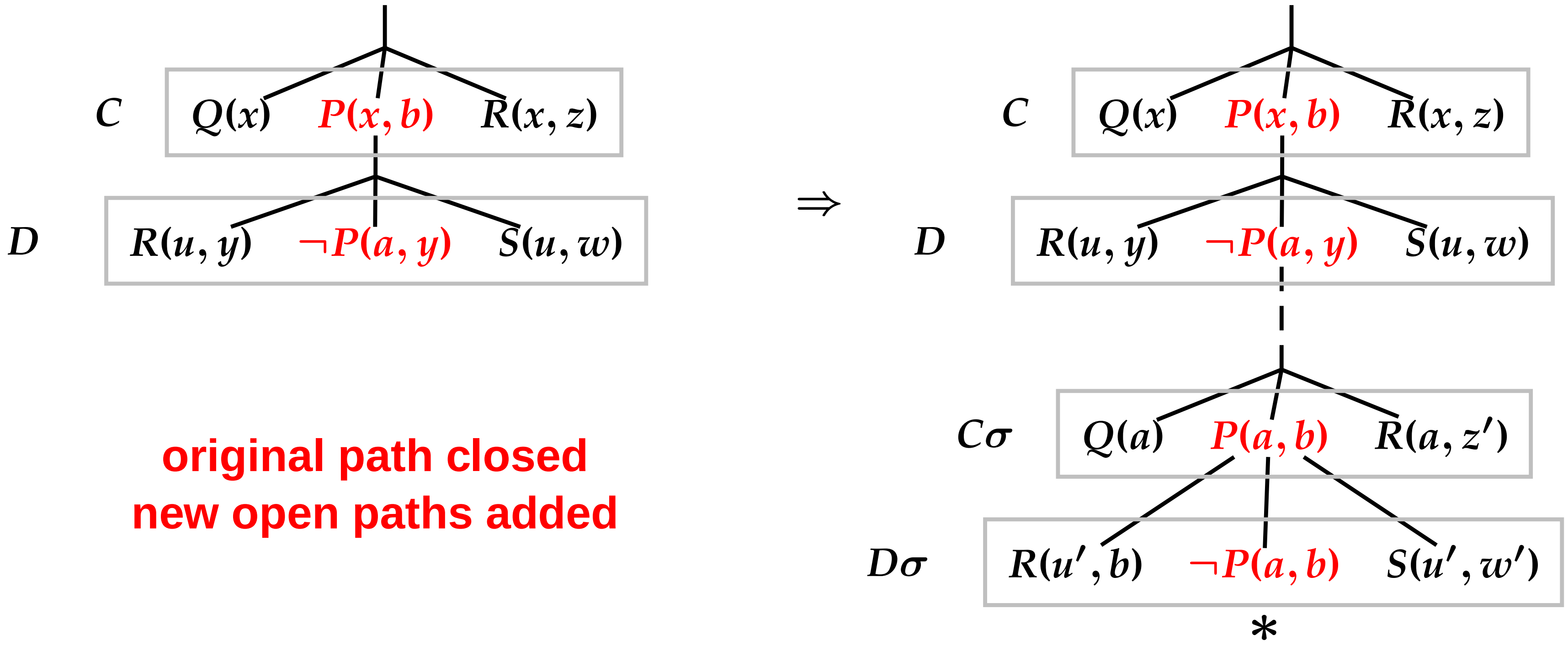




append instances with substitution $\{x/a, y/b\}$ to path

The Disconnection Calculus (II)

- Singular inference rule: **Linking**



- Concept of $\forall$-closure of branches

  closure by simultaneous instantiation of all variables by the same constant: path with $P(x,y)$ and $\neg P(z,z)$ is closed

Proof Search in the Disconnection Calculus

- **Proof process in two phases:**
  - **An initial active path through the formula is don’t-care nondeterministically selected**
  - **Using the links contained in the active path, instances of linked clauses are used to build a tableau**
- **An open tableau path may be selected don’t-care nondeterministically, it becomes the next active path**
- **Each link can be used only once on a path (explains the name "disconnection")**
- **Absence of usable links (saturation of a path) indicates satisfiability of the formula**
- **Only requirement for (strong) completeness: fairness of link selection**

An Example Proof

Input

Clauses

$$P(x,z) \lor \neg P(x,y) \lor \neg P(y,z)$$

$$P(b,c)$$

$$P(a,b)$$

$$\neg P(a,c)$$

An Example Proof

Input

Clauses

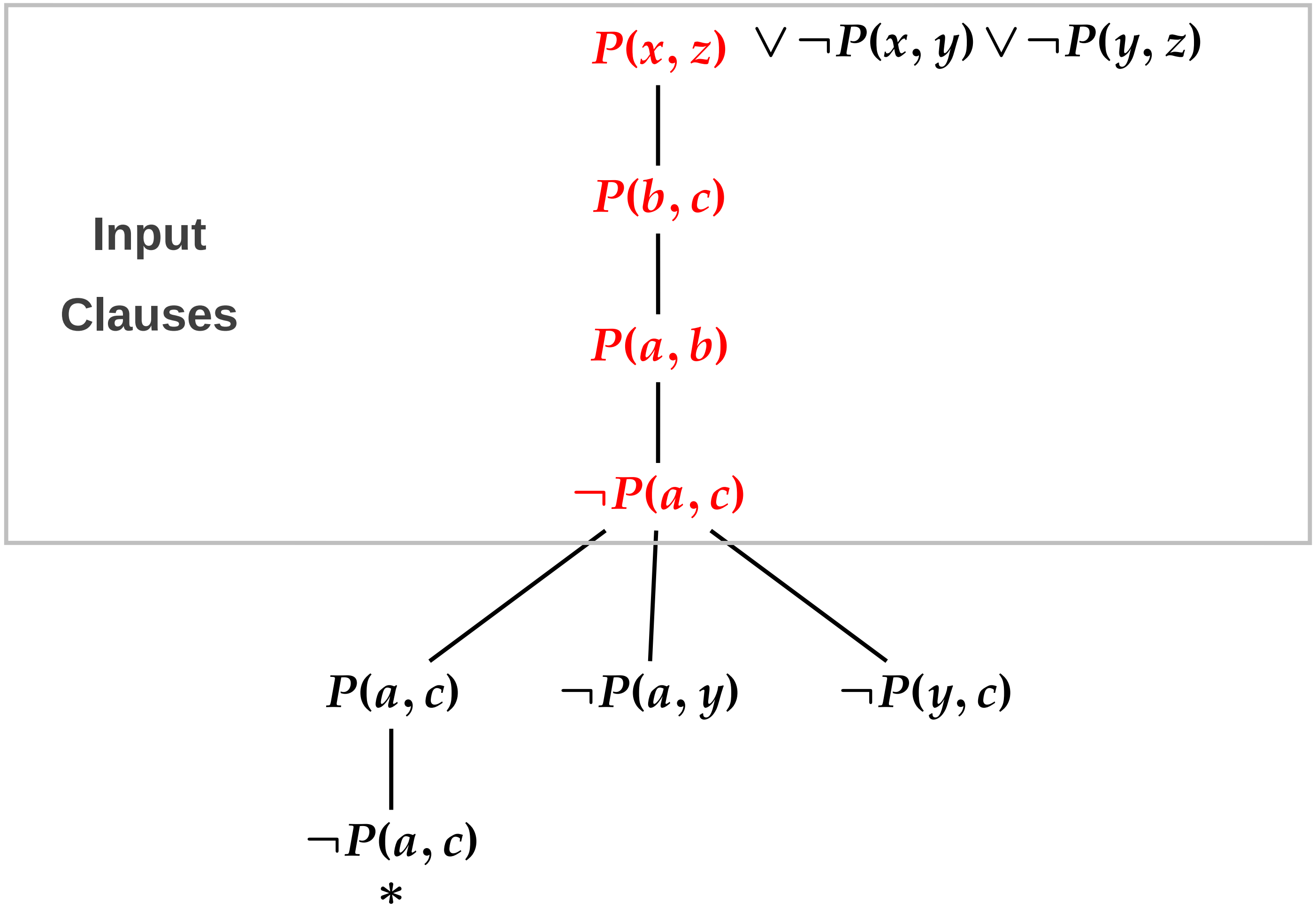
$P(x, z) \vee \neg P(x, y) \vee \neg P(y, z)$

$P(b, c)$

$P(a, b)$

$\neg P(a, c)$

An Example Proof



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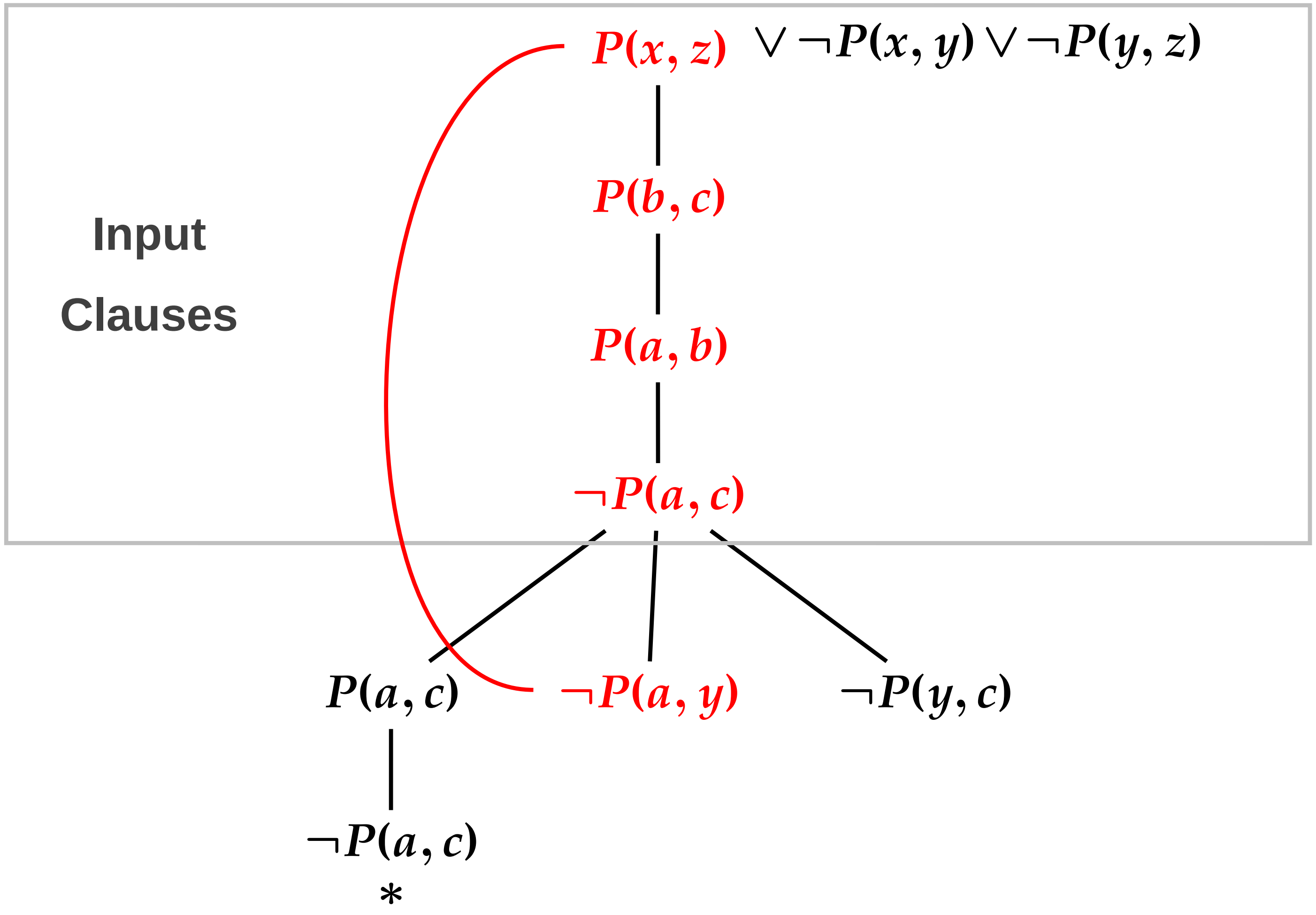

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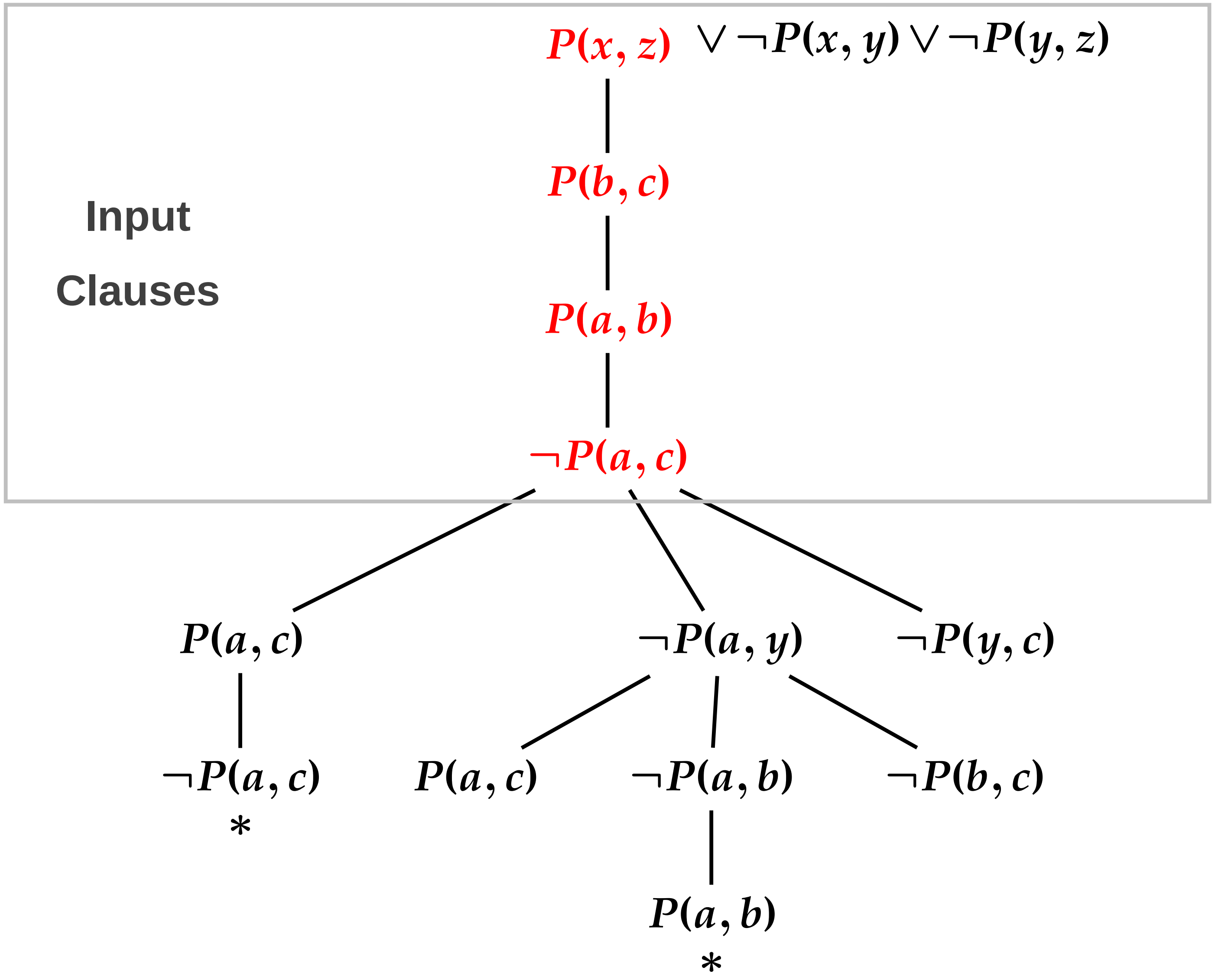

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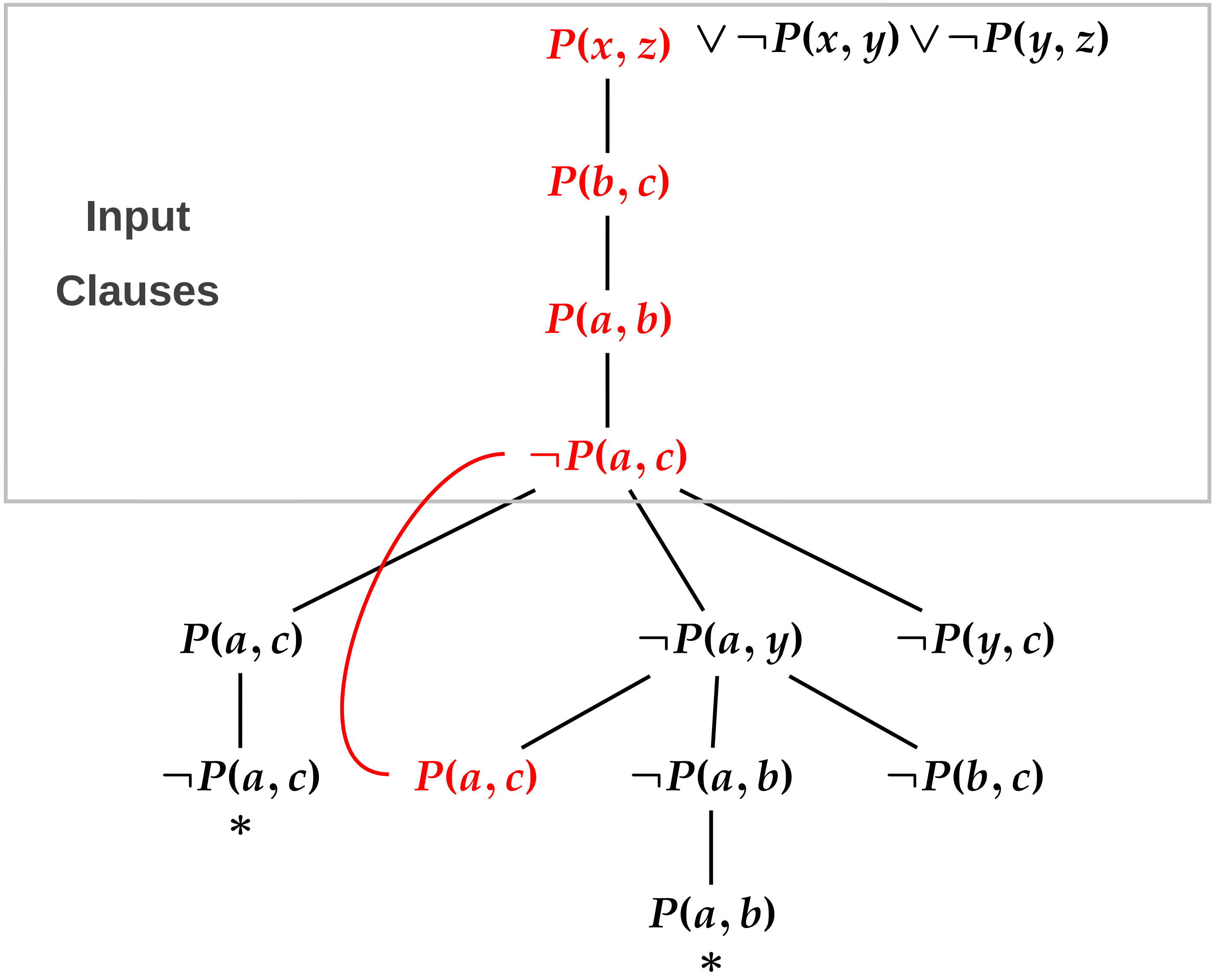

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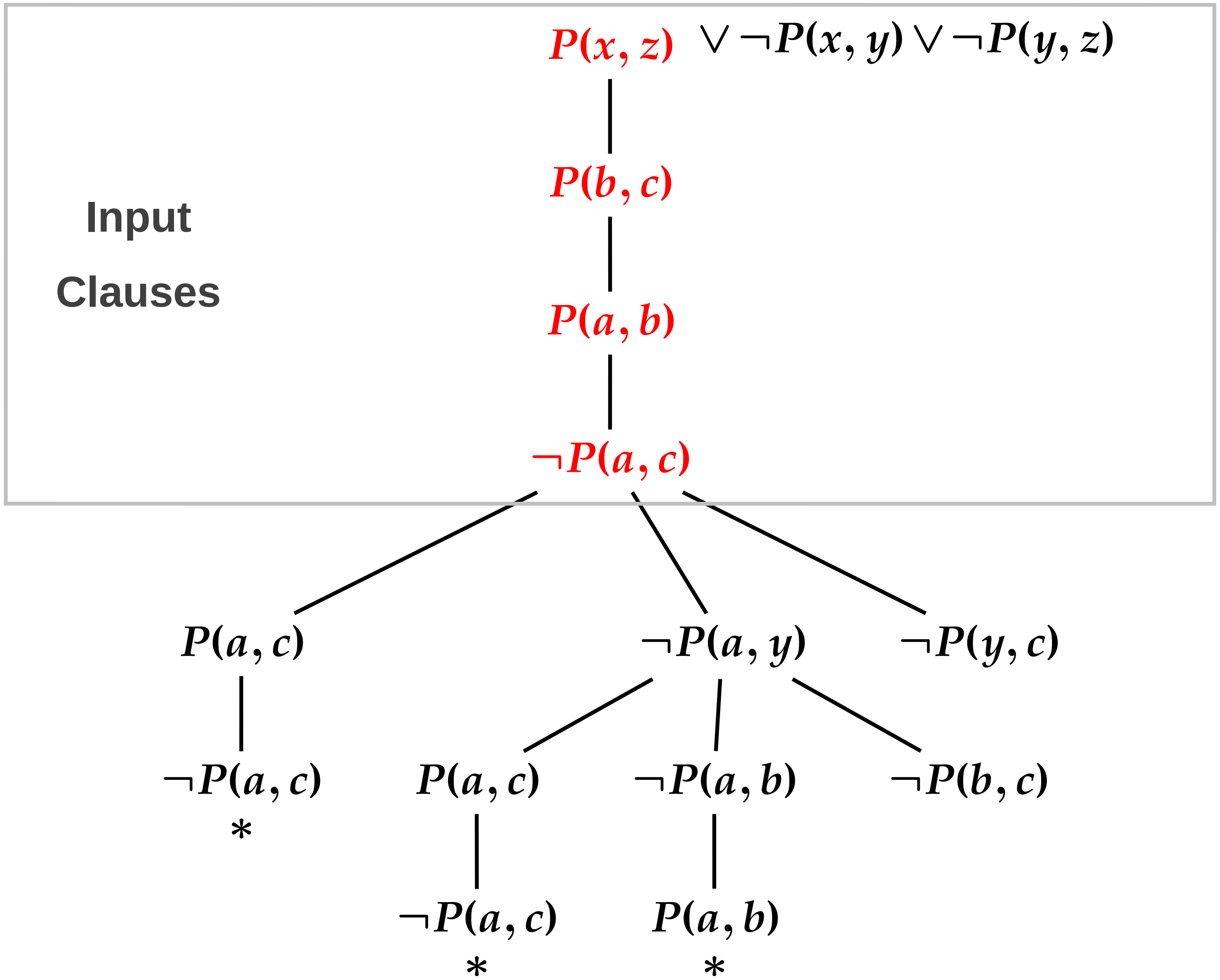

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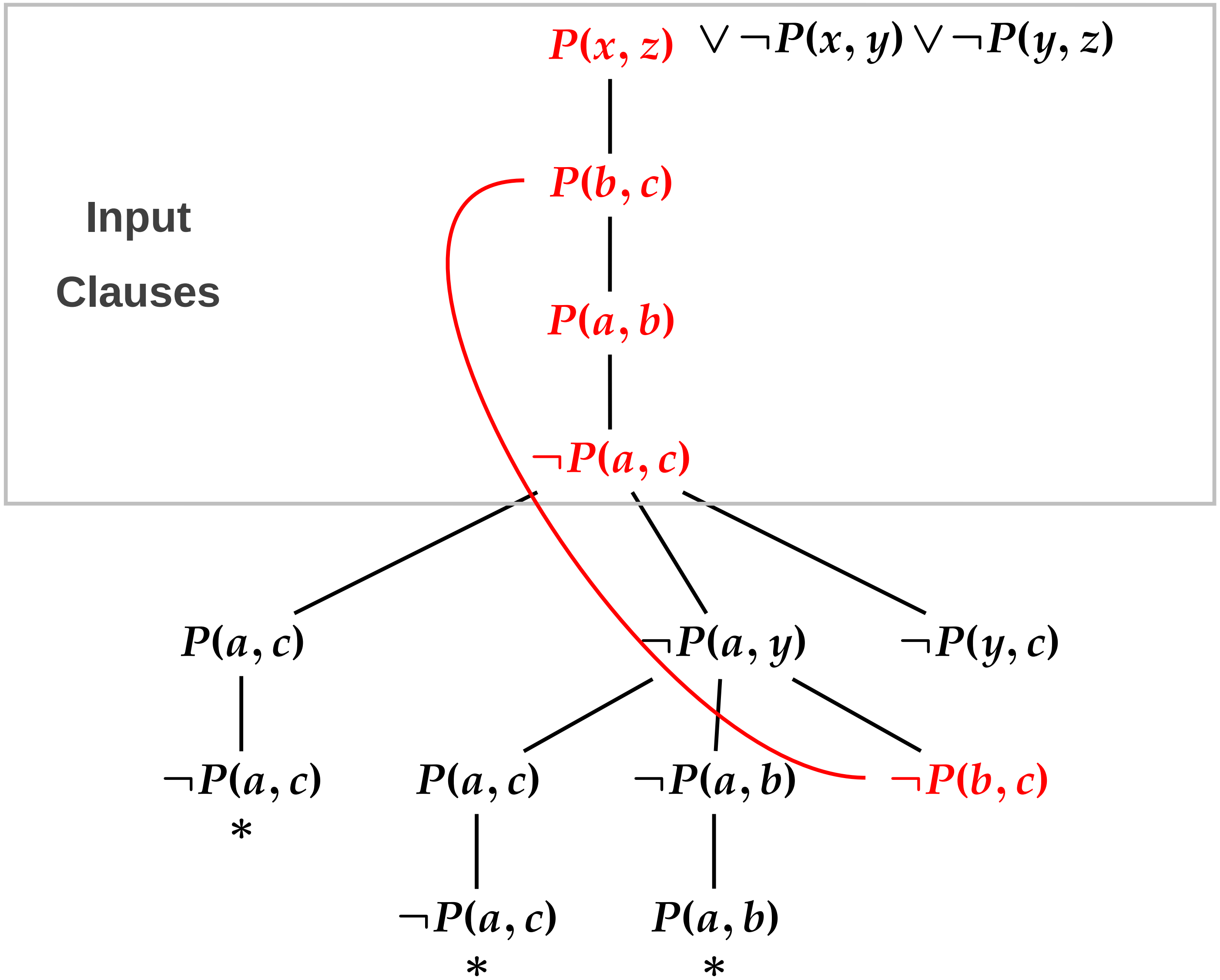

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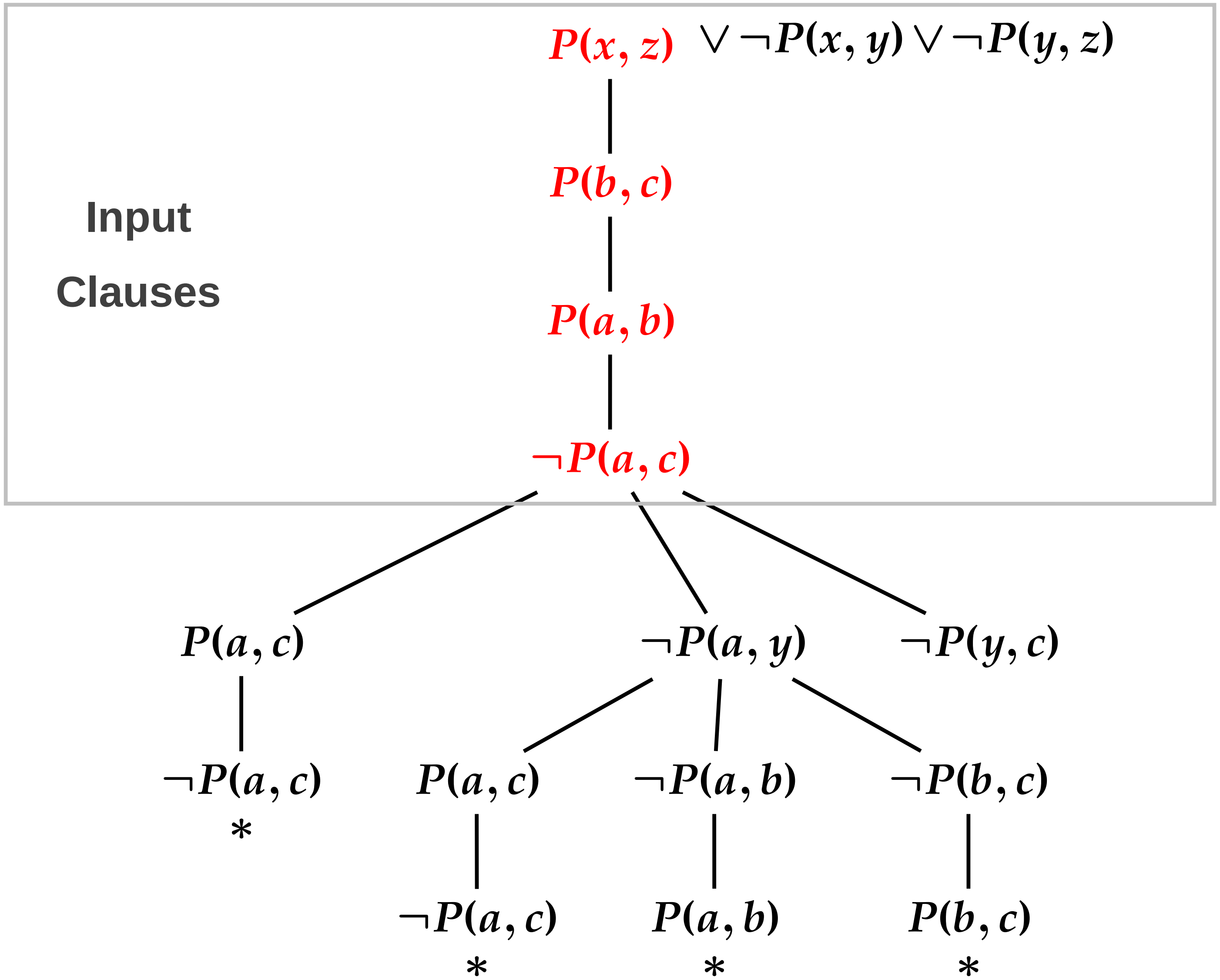

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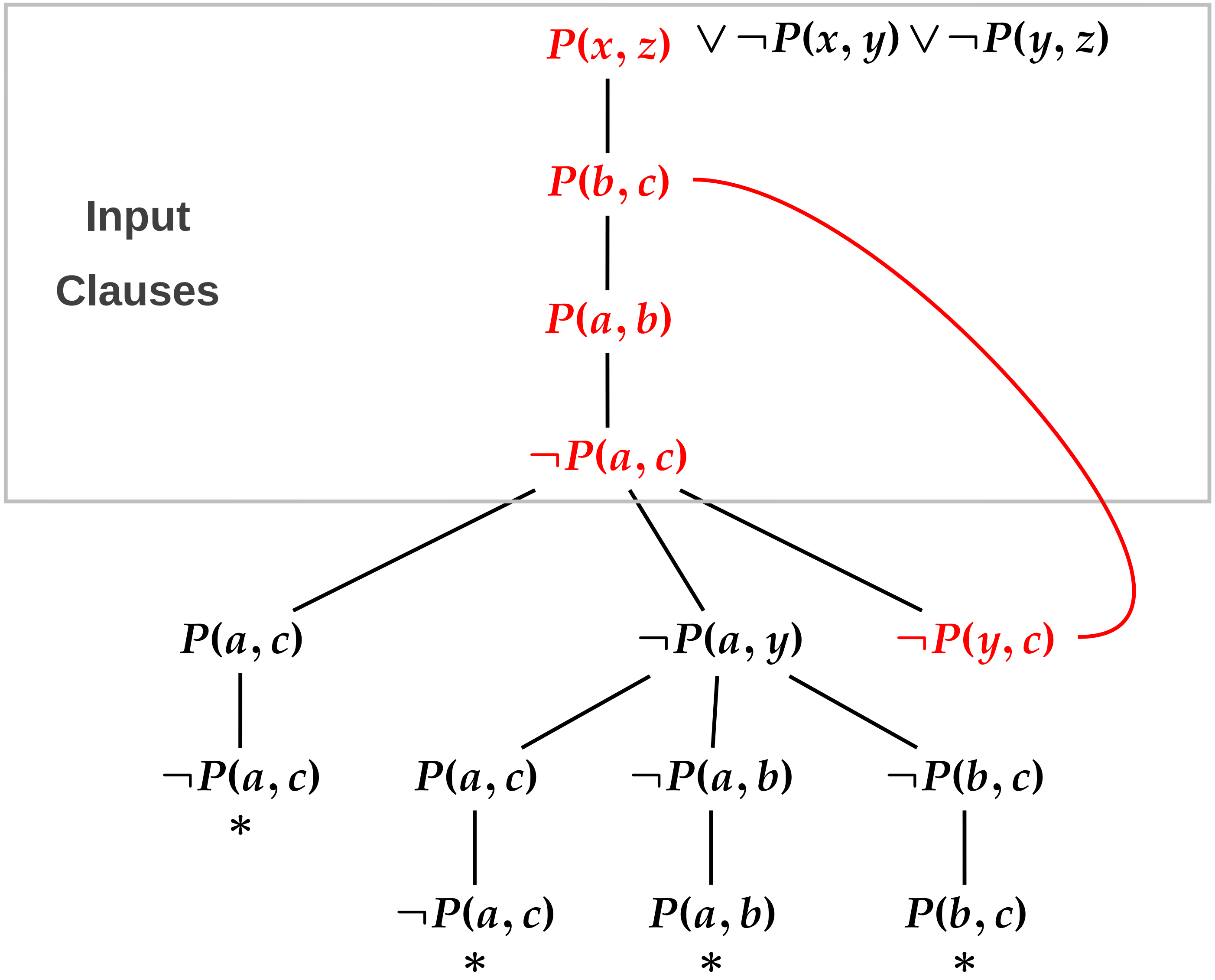

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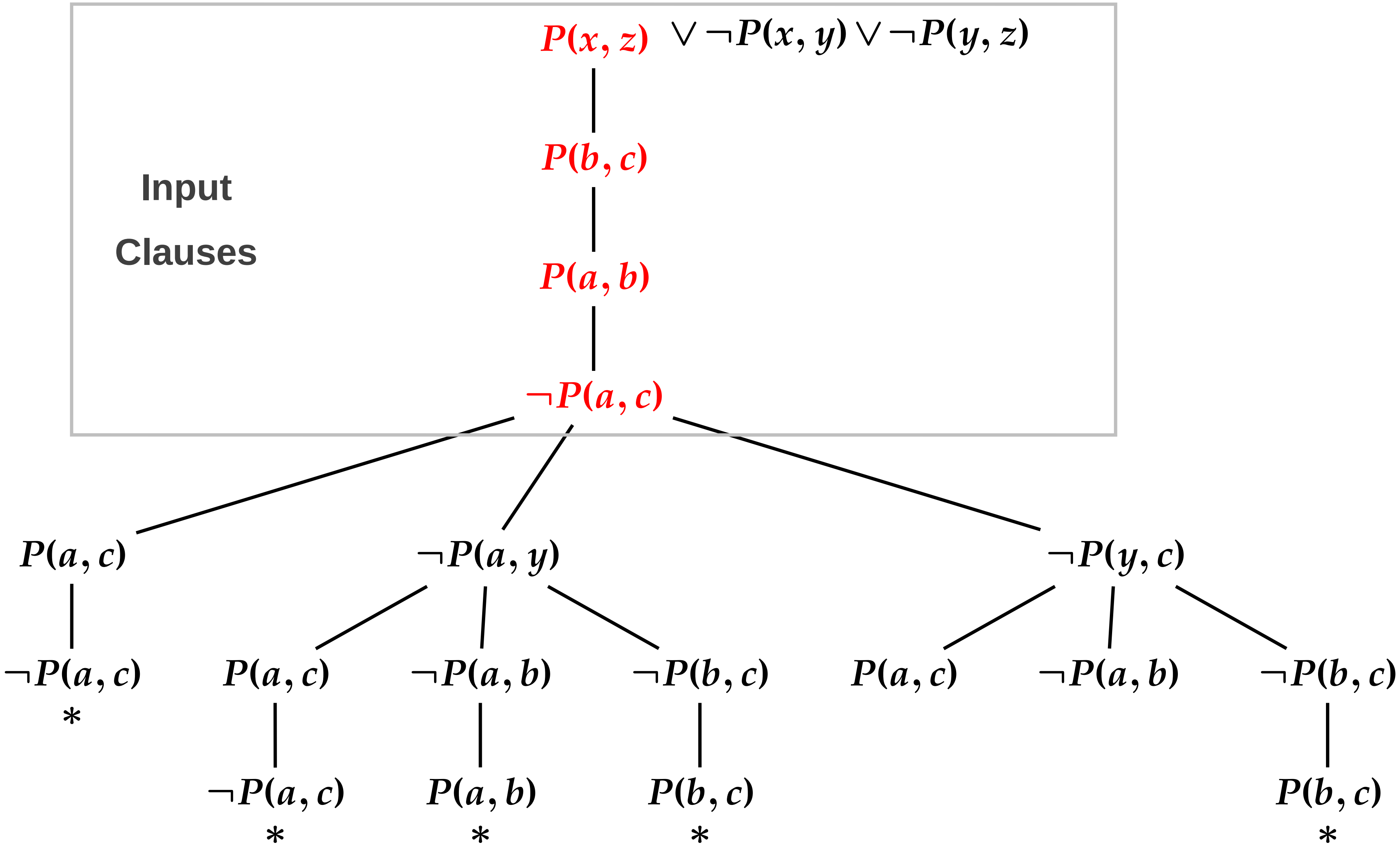

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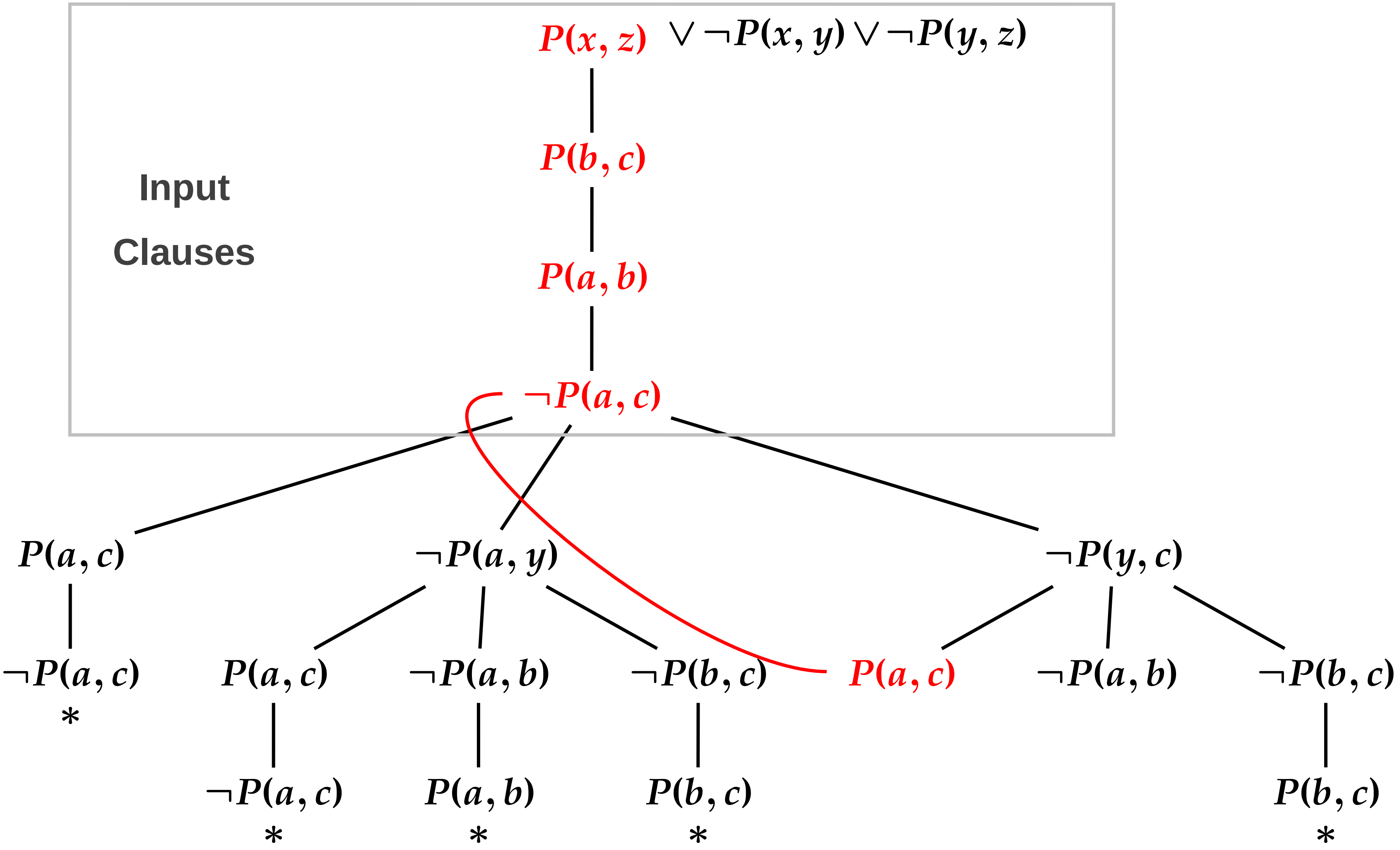

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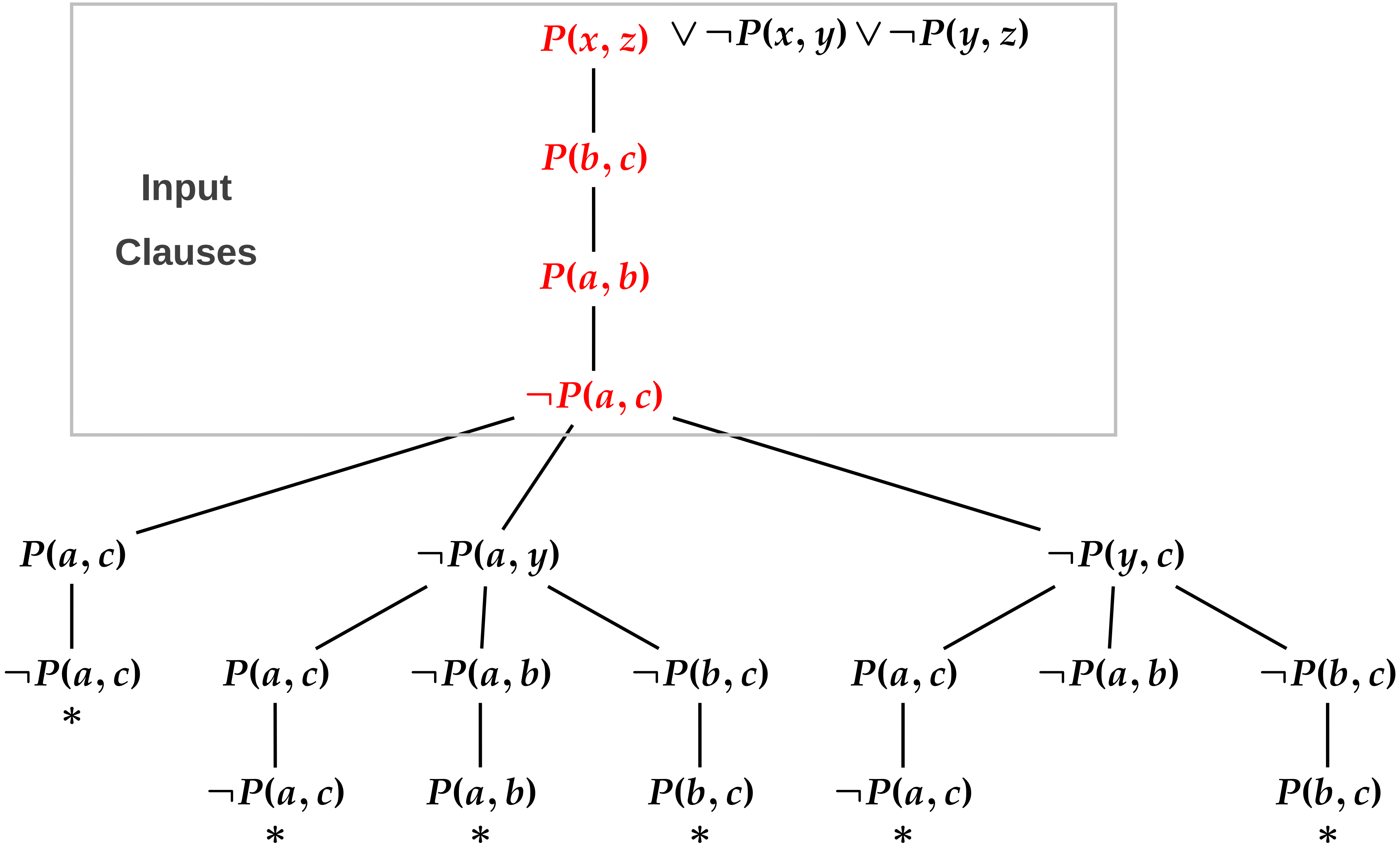

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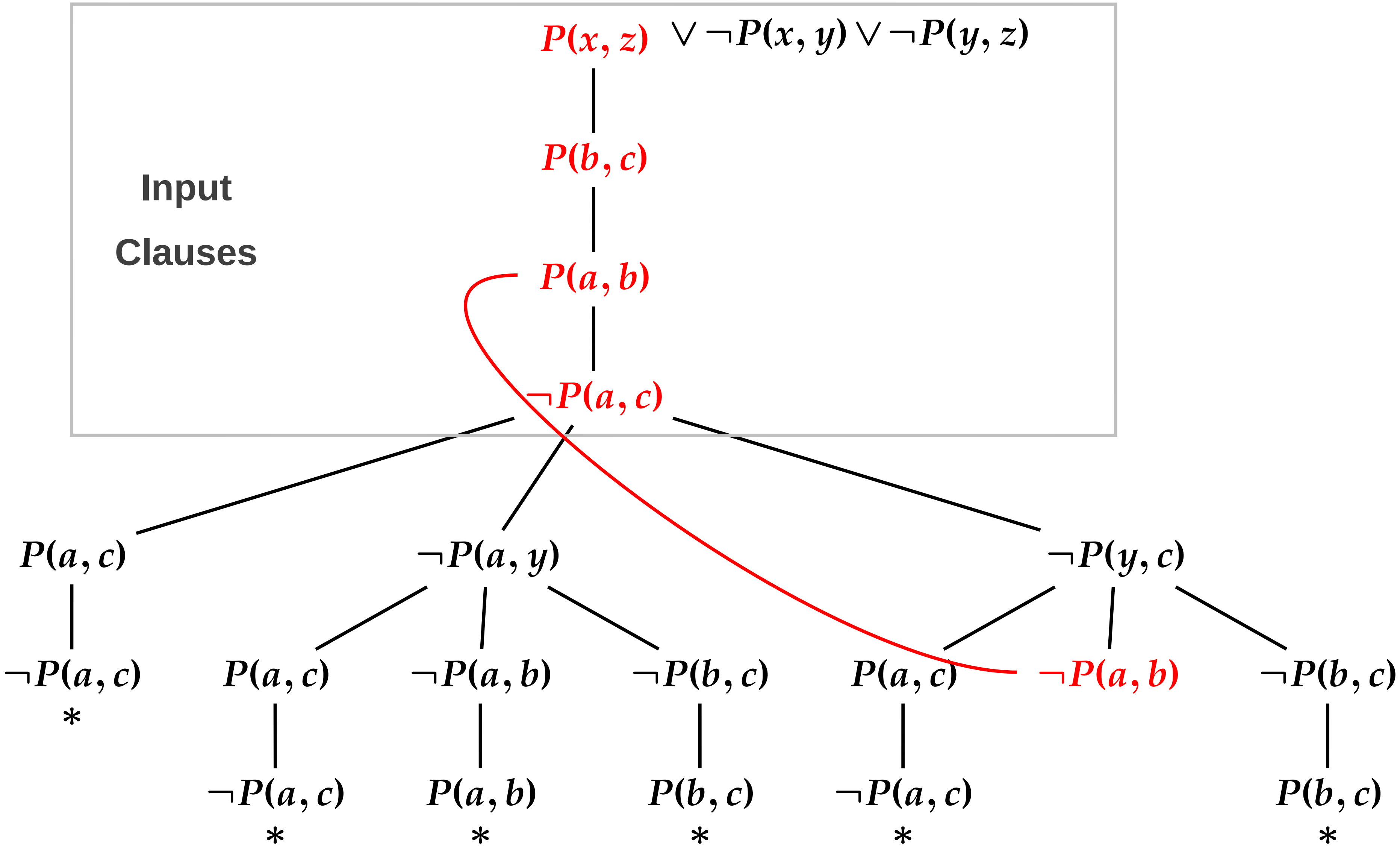

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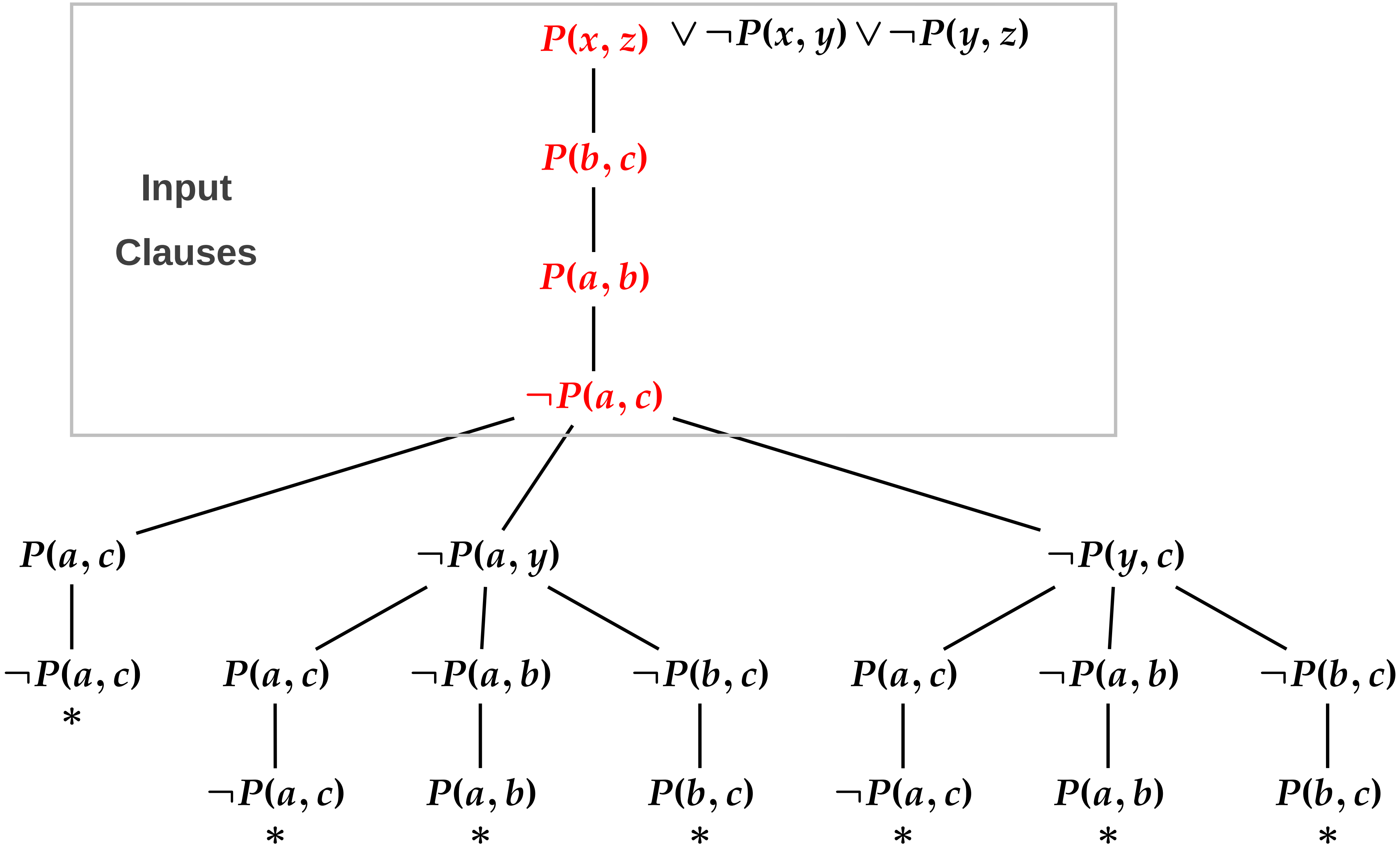

Variant Freeness

- **Two clauses are *variants* if they can be obtained from each other by variable renaming**
- **A tableau is *variant-free* if no branch contains literals l and k where the clauses of l and k are variants**
- **All disconnection tableaux are required to be variant-free**
- **Variant-freeness provides essential pruning (weak form of subsumption)**
- **Vital for model generation**
- **Implies the idea of *branch saturation*:**

  A branch is *saturated* if it cannot be extended in a variant-free manner

Failed Proof Attempts

- **Proof attempts may fail - what happens then?**

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- In order to show this, we will change one clause in the previous example: the signs are inverted

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Input Clauses

$$\neg P(x,z) \vee P(x,y) \vee P(y,z)$$

$$P(b,c)$$

$$P(a,b)$$

$$\neg P(a,c)$$

Failed Proof Attempts

- **Proof attempts may fail - what happens then?**
- **In order to show this, we will change one clause in the previous example: the signs are inverted**

Input Clauses

$$\neg P(x,z) \vee P(x,y) \vee P(y,z)$$

$$P(b,c)$$

$$P(a,b)$$

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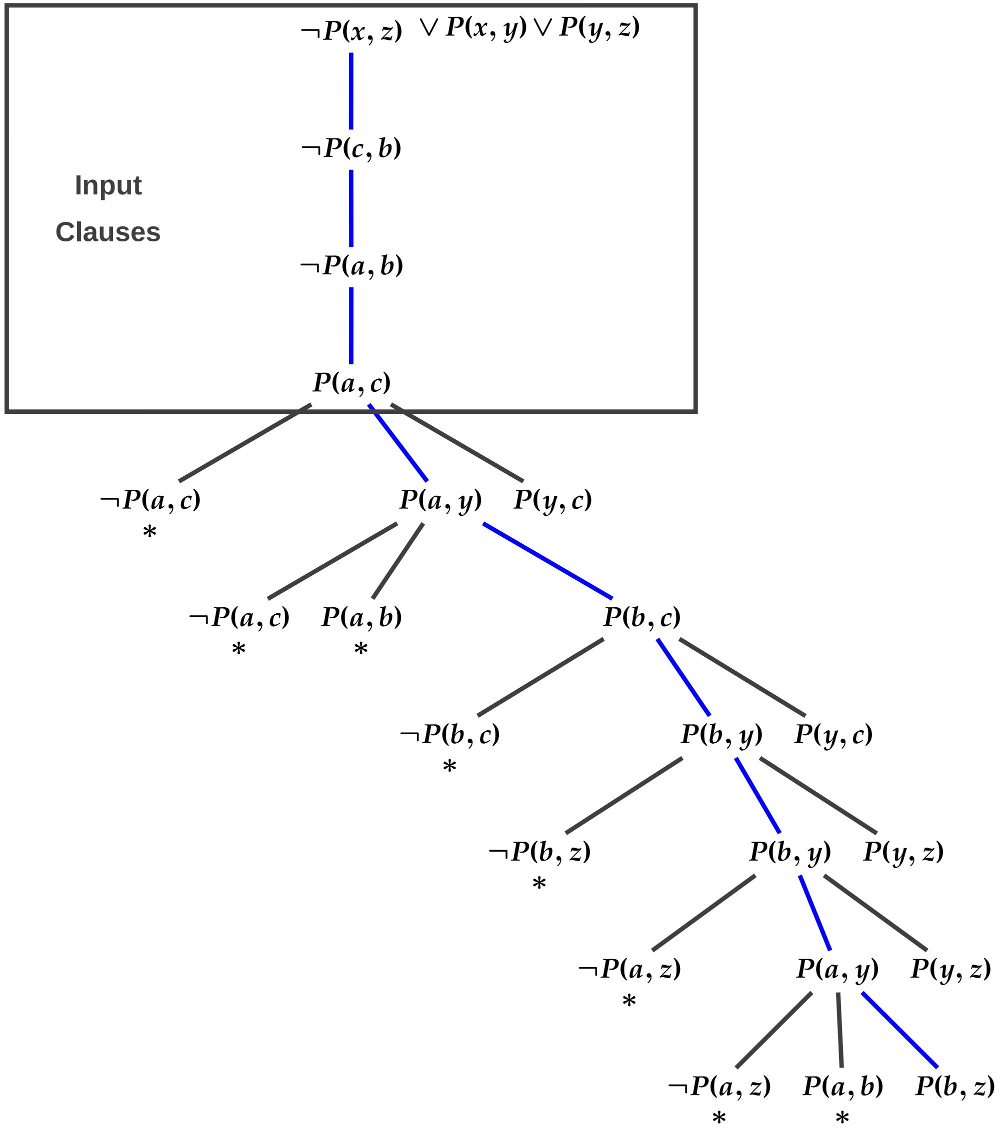
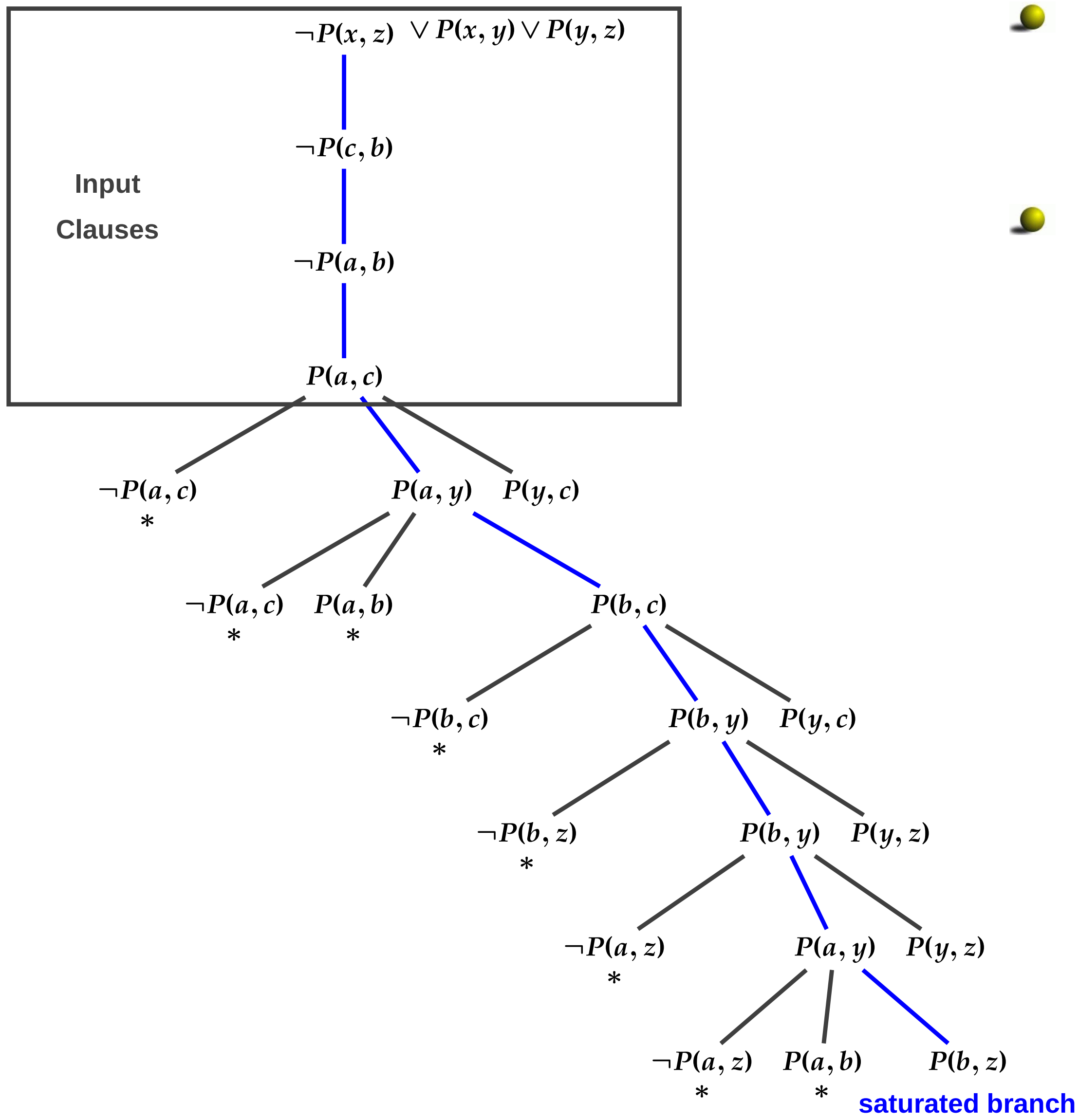
- **Again, we attempt to find a proof**

A Saturated Open Tableau



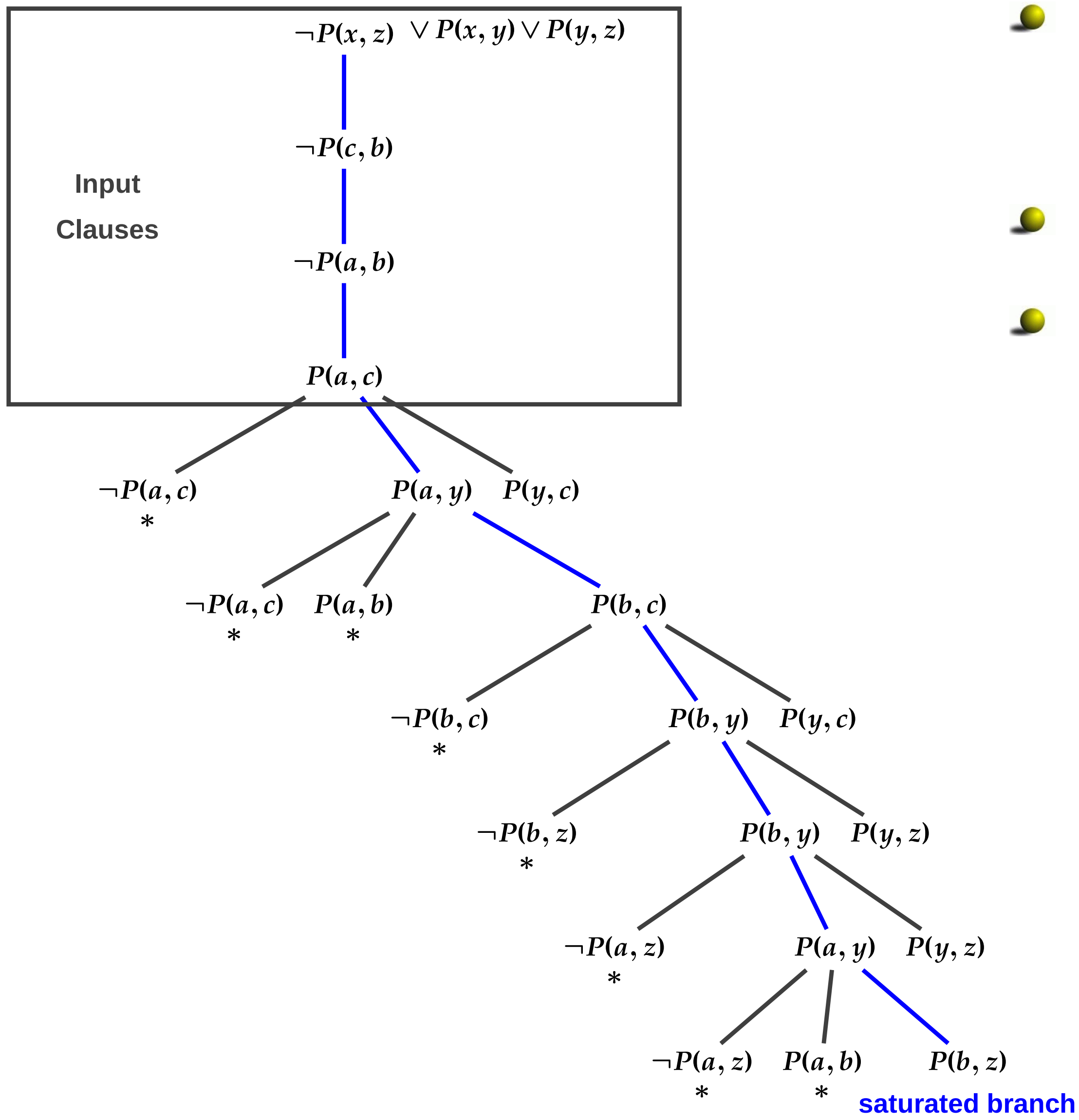
- **This open tableau cannot be closed**

A Saturated Open Tableau



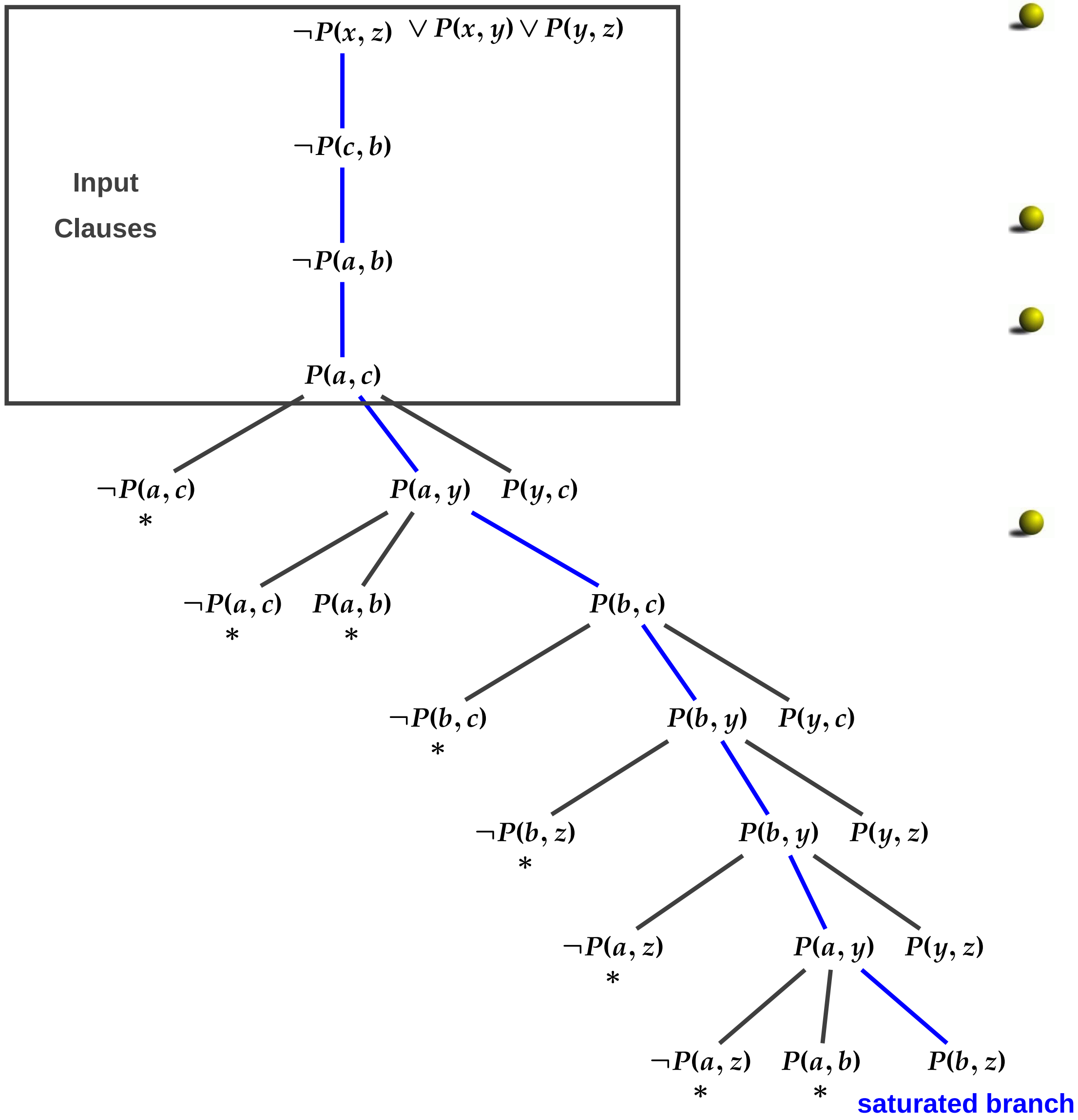
- **This open tableau cannot be closed**
- **Indicated branch is saturated**

A Saturated Open Tableau



- This open tableau cannot be closed
- Indicated branch is saturated
- Saturated open branch provides model

A Saturated Open Tableau



- **This open tableau cannot be closed**
- **Indicated branch is saturated**
- **Saturated open branch provides model**
- **How to extract model?**

Instance Preserving Enumerations

- *Instance Preserving Enumerations*: lists of literal occurrences on a path
- Path literals are partially ordered in enumeration (not unique)
- Each literal must occur before all more general instances of itself
- Instance preserving enumeration of a saturated open branch implies model
- Example: For the open (sub-) branch

$$\begin{array}{c} \neg P(a) \\ | \\ P(x) \\ | \\ \neg P(c) \end{array}$$

With Herbrand universe $\{a, b, c, d, e\}$ and enumeration

$$[\neg P(a) \quad \neg P(c) \quad P(x)]$$

the model implied is $\{\neg P(a), P(b), \neg P(c), P(d), P(e)\}$

Model Extraction

We extract an instance preserving enumeration for the open branch of the preceding tableau:

$$\boxed{\neg P(c,b)}$$

$$\boxed{\neg P(a,b)}$$

$$\boxed{P(a,c)}$$

$$\neg P(a,c) \vee P(a,b) \vee \boxed{P(b,c)}$$

$$\neg P(a,z) \vee P(a,b) \vee \boxed{P(b,z)}$$

$$\neg P(a,c) \vee \boxed{P(a,y)} \vee P(y,c)$$

$$\neg P(b,c) \vee \boxed{P(b,y)} \vee P(y,c)$$

$$\neg P(b,z) \vee \boxed{P(b,y)} \vee P(y,z)$$

$$\neg P(a,z) \vee \boxed{P(a,y)} \vee P(y,z)$$

$$\boxed{\neg P(x,z)} \vee P(x,y) \vee P(y,z)$$

Model Extraction

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$$\boxed{\neg P(a,b)}$$
$$\boxed{P(a,c)}$$
$$\neg P(a,c) \lor P(a,b) \lor \boxed{P(b,c)}$$
$$\neg P(a,z) \lor P(a,b) \lor \boxed{P(b,z)}$$
$$\neg P(a,c) \lor \boxed{P(a,y)} \lor P(y,c)$$
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$$\neg P(a,z) \lor \boxed{P(a,y)} \lor P(y,z)$$
$$\boxed{\neg P(x,z)} \lor P(x,y) \lor P(y,z)$$

From which we get the finite Herbrand model:

$$\{ \quad \neg P(c,b), \neg P(a,b), P(a,c), \\ P(b,c), P(b,a), P(b,b), \\ P(a,a), \neg P(c,a), \neg P(c,c) \quad \}$$

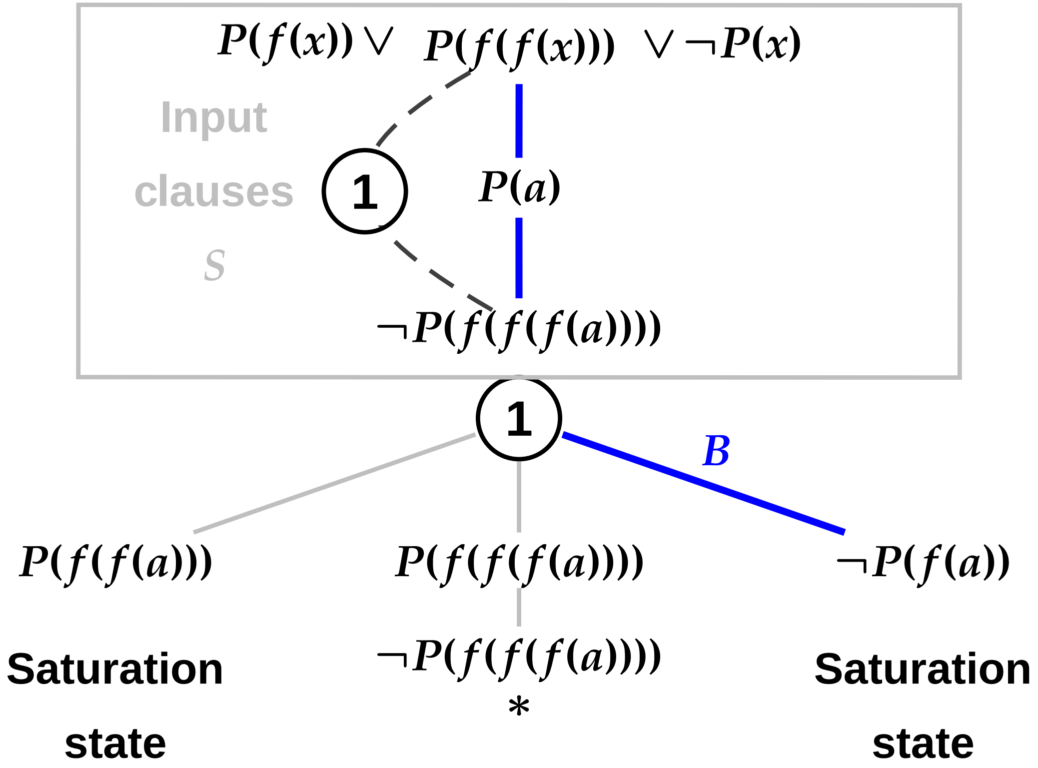
Infinite Herbrand Models

Model extraction also works for infinite Herbrand universes

Infinite Herbrand Models

Model extraction also works for infinite Herbrand universes

Given a saturated tableau with open branch B:



Infinite Herbrand Models

Model extraction also works for infinite Herbrand universes

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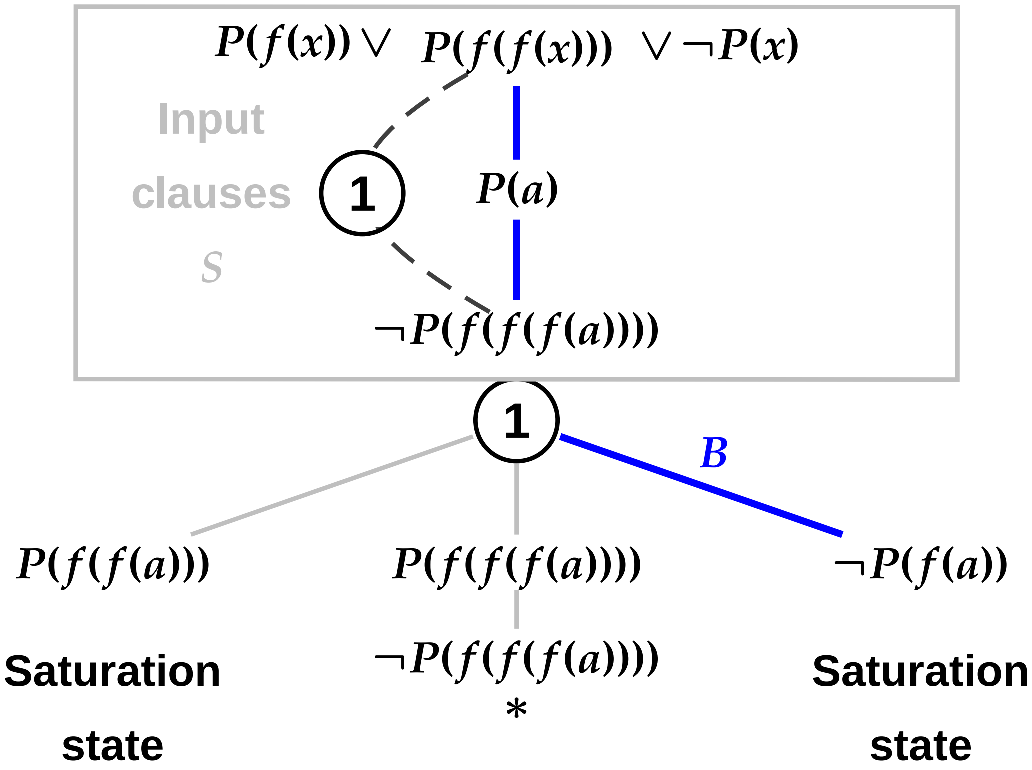


The enumeration for B

$\neg P(f(f(f(a))))$, $\neg P(f(a))$, $P(a)$, $P(f(f(x)))$

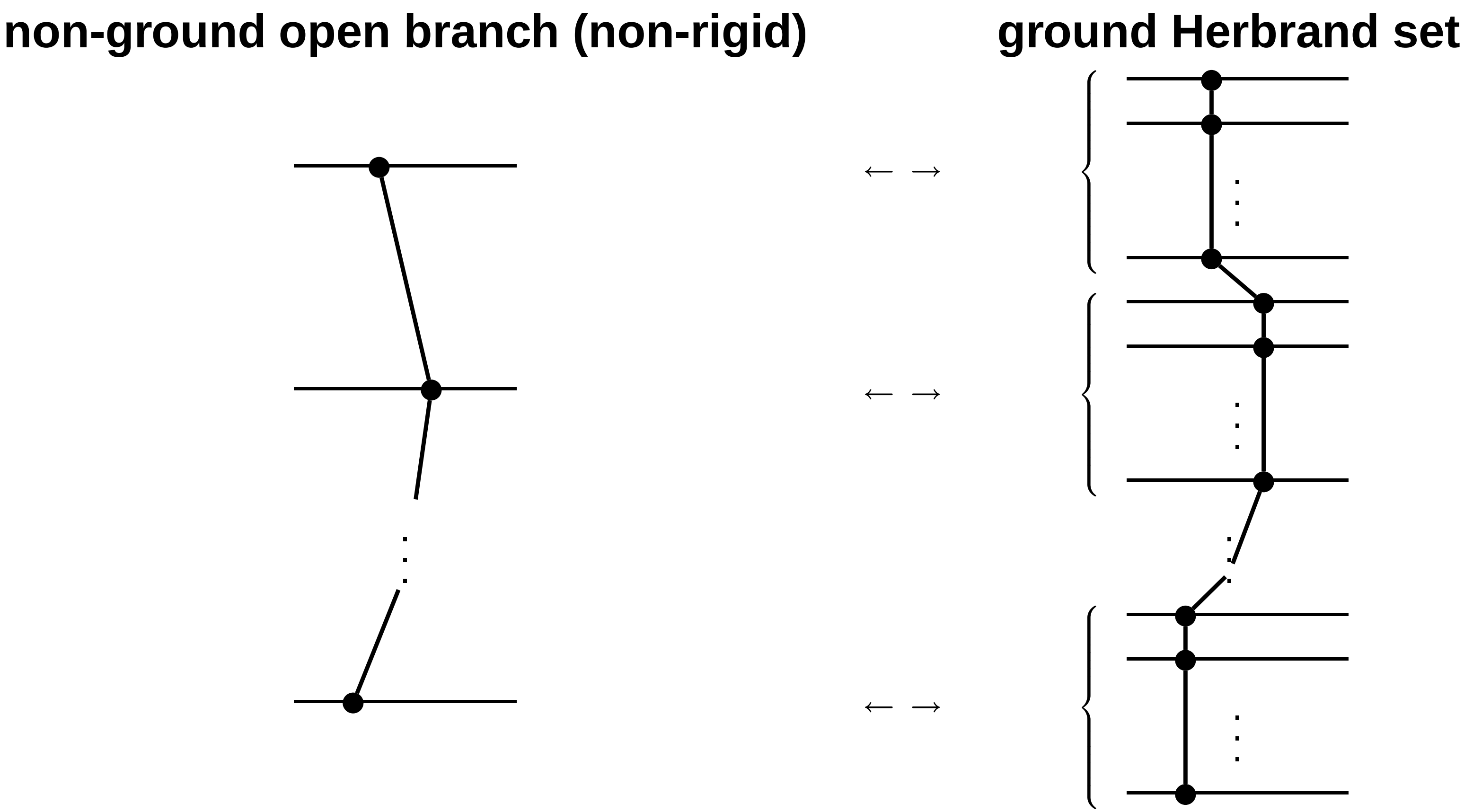
implies a finite representation of an infinite Herbrand model:

$\{\neg P(f(f(f(a)))), \neg P(f(a)), P(a)\}, \{P(f(f(\mathfrak{s})))\}$

with the constraint $\mathfrak{s} \neq f(a)$, where $\mathfrak{s}$ ranges over the Herbrand universe of S.

Completeness

- **Basic concept: open saturated branch represents partial model**
- **Non-equational case: branch determines path through Herbrand set**

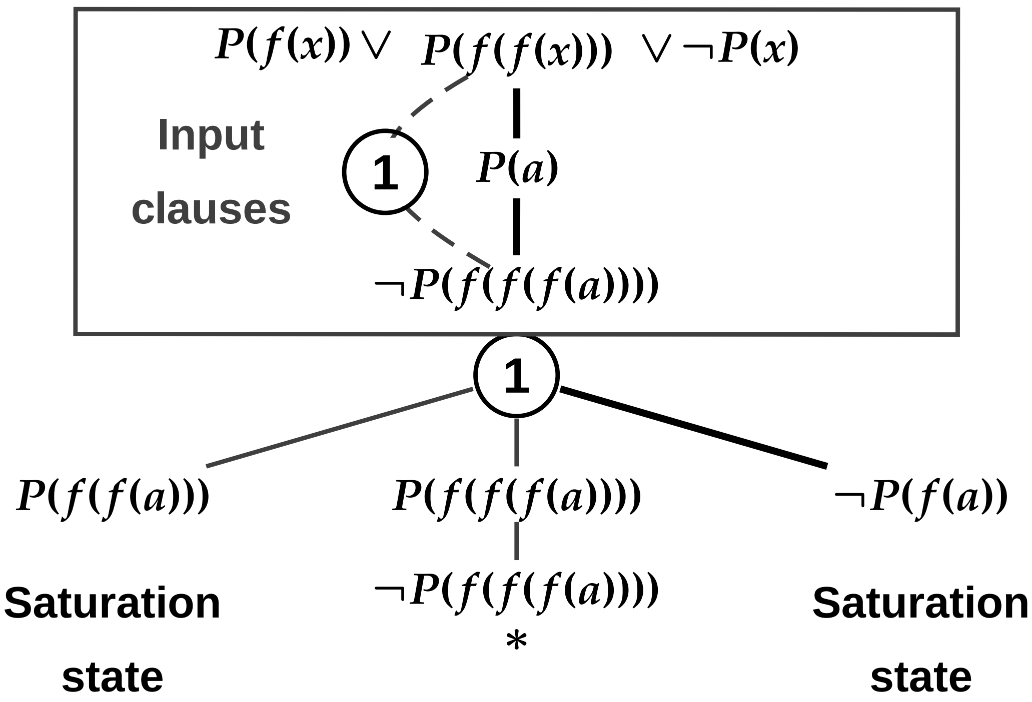


- **Closed ground path corresponds to applicable link**

  $\Leftrightarrow$ contradicts saturation

The Saturation Property

- Saturated open branch specifies a model (only such a branch)
- Model characterised as exception-based representation (EBR)



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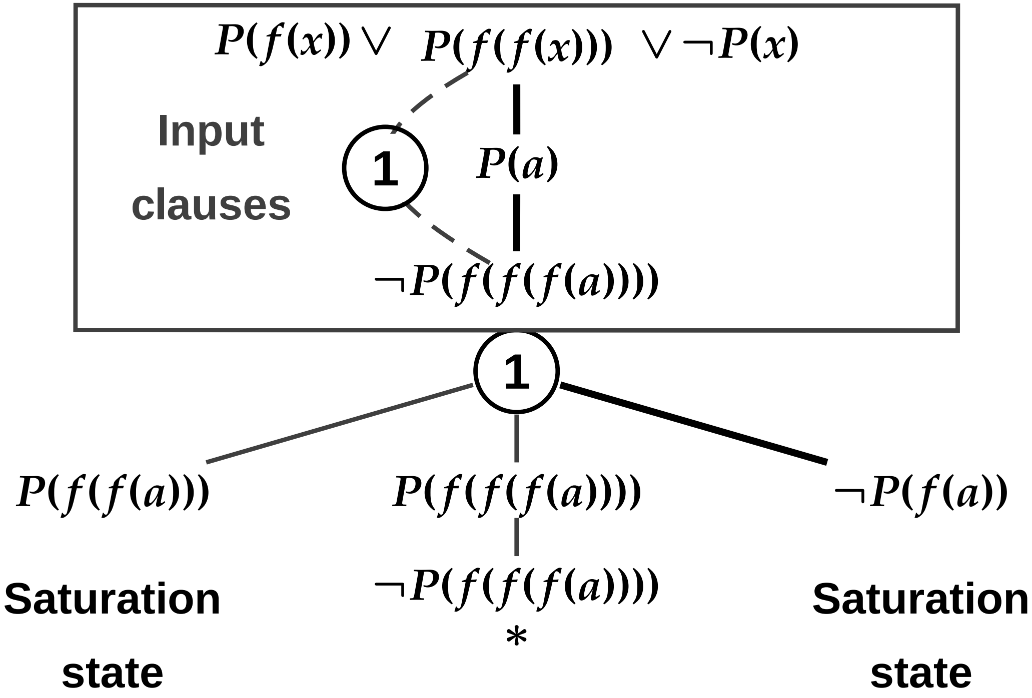

- **Model:** $\{\neg P(f(f(f(a)))), \neg P(f(a)), P(a)\} \cup \{P(f(f(\mathfrak{s}))) : \mathfrak{s} \neq f(a)\}$

The Saturation Property

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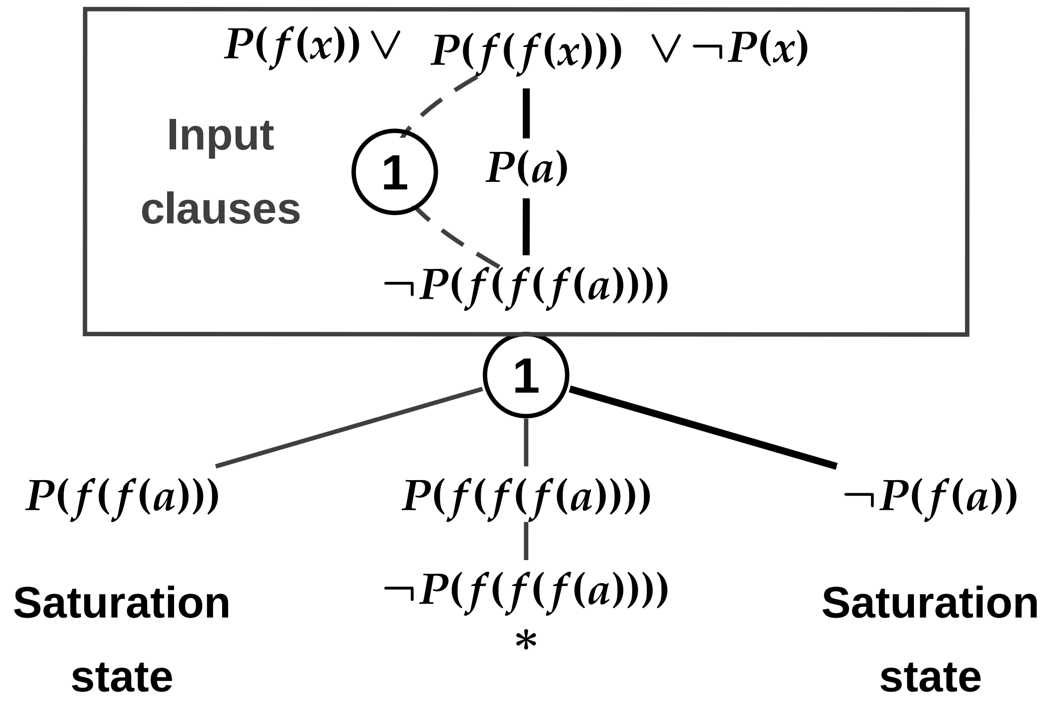
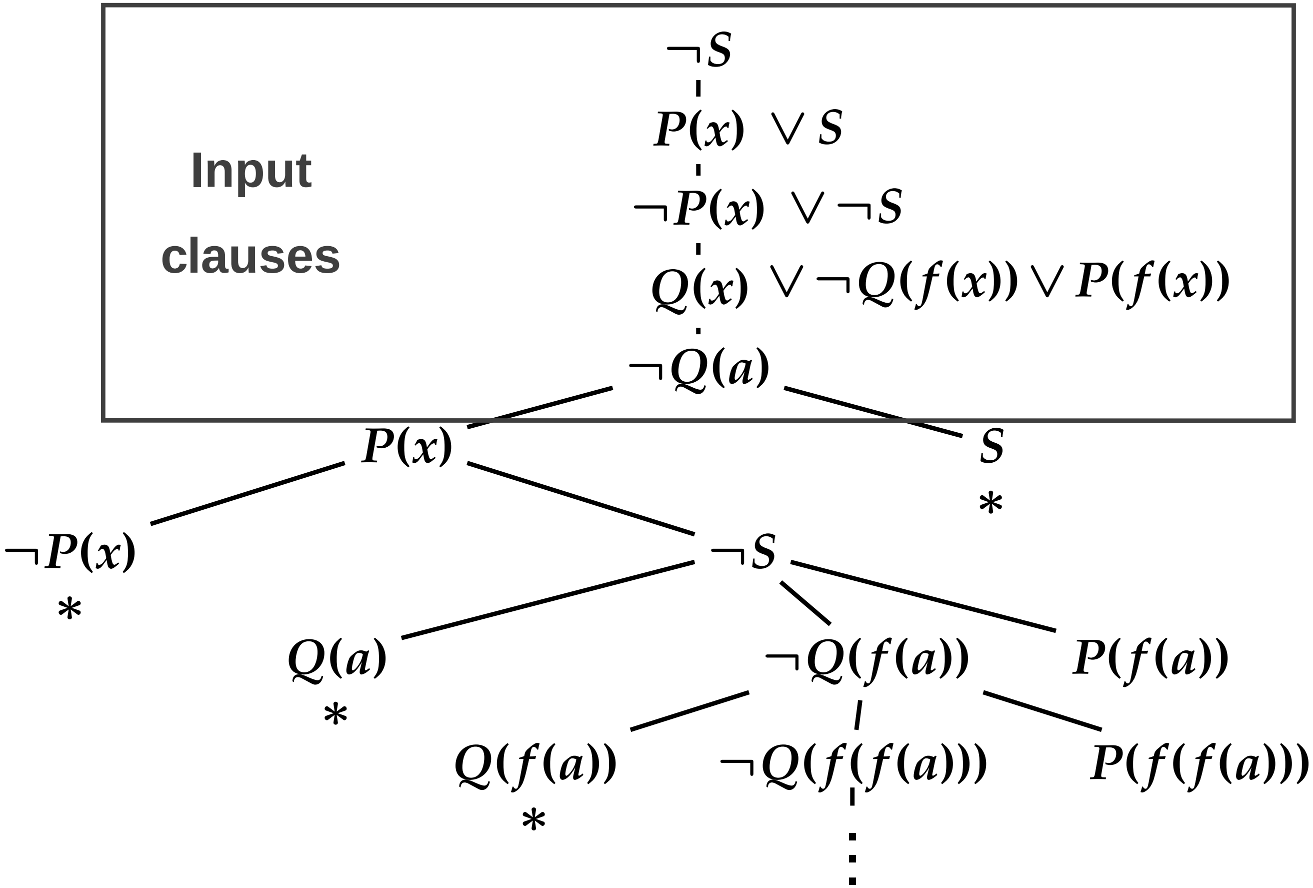


- EBR for model: $\{P(a), \neg P(f(a)), P(f(f(x))), \neg P(f(f(f(a))))\}$

An Example for Non-Termination



- **The above problem is obviously satisfiable (P true, S and Q false)**
- **However, in general, the disconnection calculus does not terminate**
- **Termination fragile, depends on branch selection function**

The Problem

- **Here, the model is approximated, but not finitely represented**

  $\{P(x), \neg S, \neg Q(a), \neg Q(f(a)), \neg Q(f(f(a))), \neg Q(f(f(f(a))))\ldots\}$

- **Observation: linking instances are subsumed by path literal** $P(x)$
- **But: general subsumption does not work**
- **What can we do?**

Link Blocking

- **Original idea of model characterisation:**
  - **Currently considered branch is seen as an interpretation I**
  - **If a literal L is on branch, all instances of L are considered true in I**
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Link Blocking

- Original idea of model characterisation:
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- Similar to productivity restriction in ME

Candidate Models

- **Precise criteria needed to find out whether a literal is blocking**
- **EBRs are lists of branch literals partially sorted according to respective specialisation**
- **Candidate model (CM): EBR enhanced by link blockings**
- **Blockings require a modified ordering on CMs, not necessarily based on instantiation**
- **Interpretation of a literal L given by CM-matcher:**
  the rightmost literal in CM subsuming L or $\sim L$

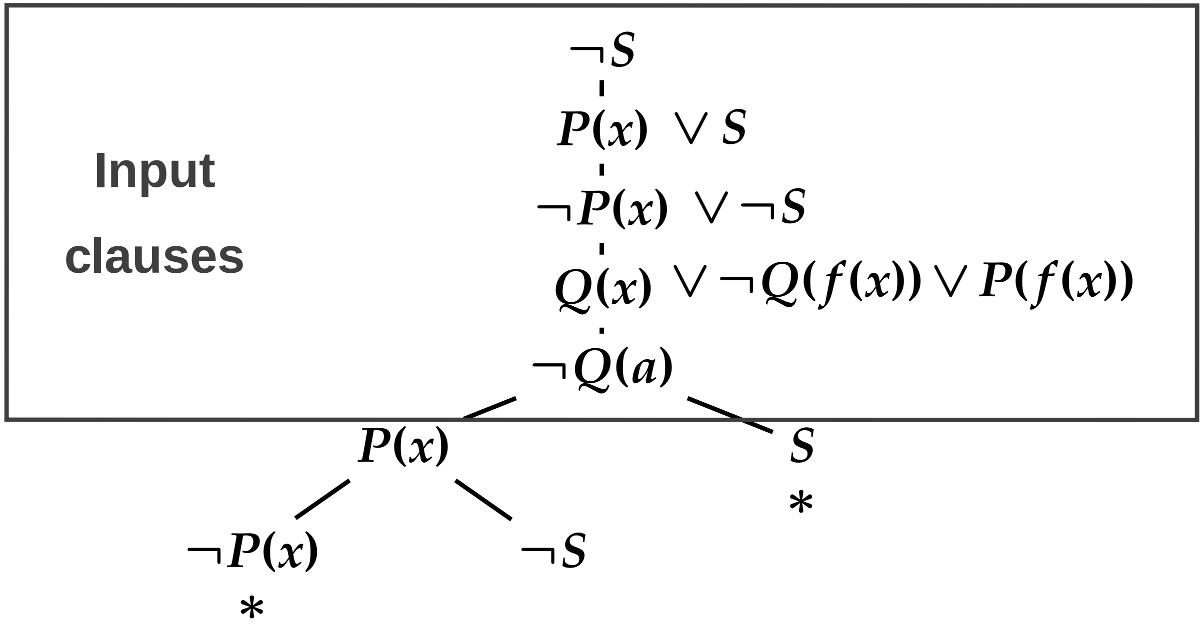
Link Blocking Example

- **The non-termination example revisited**

Input clauses	$\neg S$ $P(x) \vee S$ $\neg P(x) \vee \neg S$ $Q(x) \vee \neg Q(f(x)) \vee P(f(x))$ $\neg Q(a)$

Link Blocking Example

- **The non-termination example revisited**



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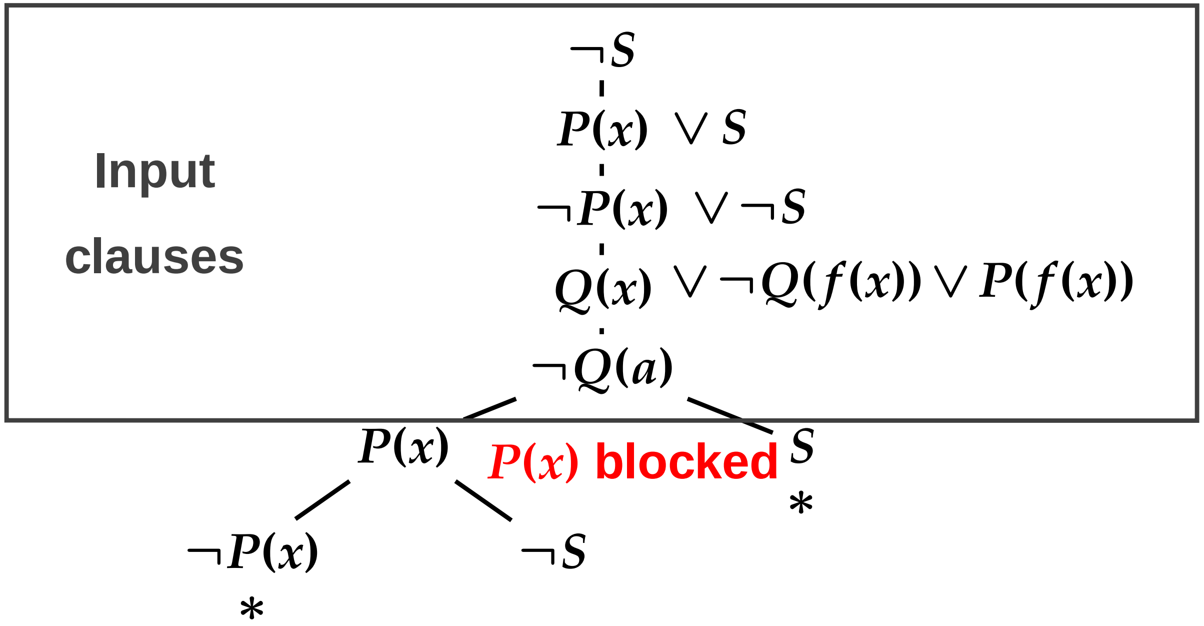

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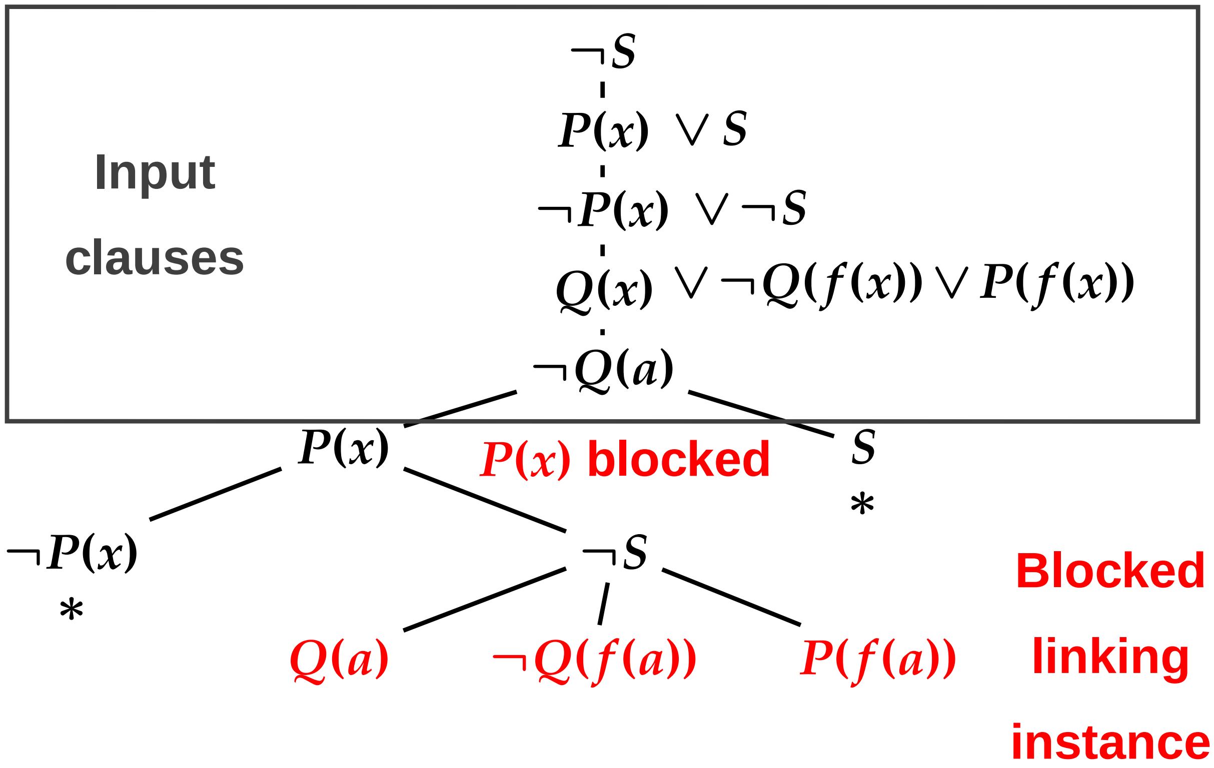

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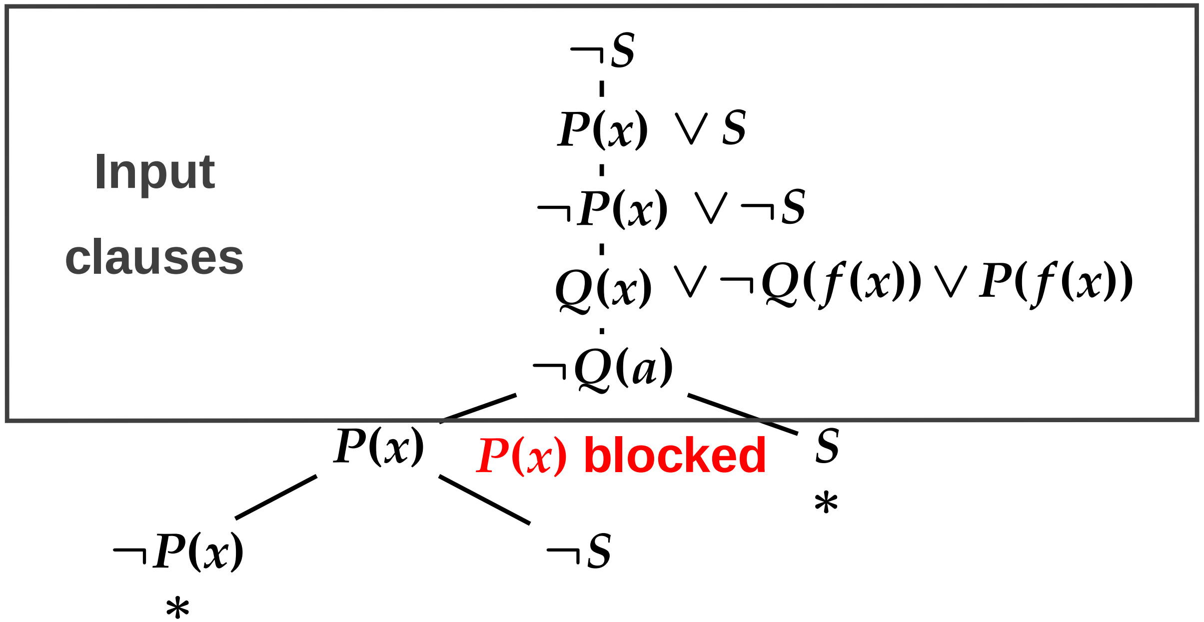

Saturation state

- Use of link blocking allows termination
- Largely independent of selection functions

Cyclic Link Blocking

$$Q(a,y) \vee \neg P(a,y)$$

$$\neg Q(x,b) \vee P(x,b)$$

$$P(a,y) \vee Q(a,y)$$

$$\neg P(x,b) \vee \neg Q(x,b)$$

Unsatisfiable clause set

Cyclic Link Blocking

two links

A

$Q(a, y) \vee \neg P(a, y)$

$\neg Q(x, b) \vee P(x, b)$

B

$P(a, y) \vee Q(a, y)$

$\neg P(x, b) \vee \neg Q(x, b)$

Unsatisfiable clause set

Cyclic Link Blocking

$Q(a, y) \vee \neg P(a, y)$

blocked **A**

$\neg Q(x, b) \vee P(x, b)$

$P(a, y) \vee Q(a, y)$

blocked **B**

$\neg P(x, b) \vee \neg Q(x, b)$

Unsatisfiable clause set

Cyclic Link Blocking

blocked A $Q(a, y) \vee \neg P(a, y)$

$\neg Q(x, b) \vee P(x, b)$

blocked B $P(a, y) \vee Q(a, y)$

$\neg P(x, b) \vee \neg Q(x, b)$

Unsatisfiable clause set

no link applicable

- For the above clause set, using blockings no refutation can be found
- Reason: The blocking relation for the clause set is cyclic
- To preserve completeness, blocking cycles must be avoided
- Well-founded ordering imposed on link blockings based on branch position

Cyclic Link Blocking Resolved

- **We try again, this time with a blocking ordering**

$Q(a, y) \vee \neg P(a, y)$

A

$\neg Q(x, b) \vee P(x, b)$

$P(a, y) \vee Q(a, y)$

B

$\neg P(x, b) \vee \neg Q(x, b)$

Unsatisfiable clause set

Cyclic Link Blocking Resolved

- **We try again, this time with a blocking ordering**

		$Q(a,y) \vee \neg P(a,y)$	
not blocked	A	|	
		$\neg Q(x,b) \vee P(x,b)$	Unsatisfiable
		|	
		$P(a,y) \vee Q(a,y)$	clause set
blocked	B	|	
		$\neg P(x,b) \vee \neg Q(x,b)$	

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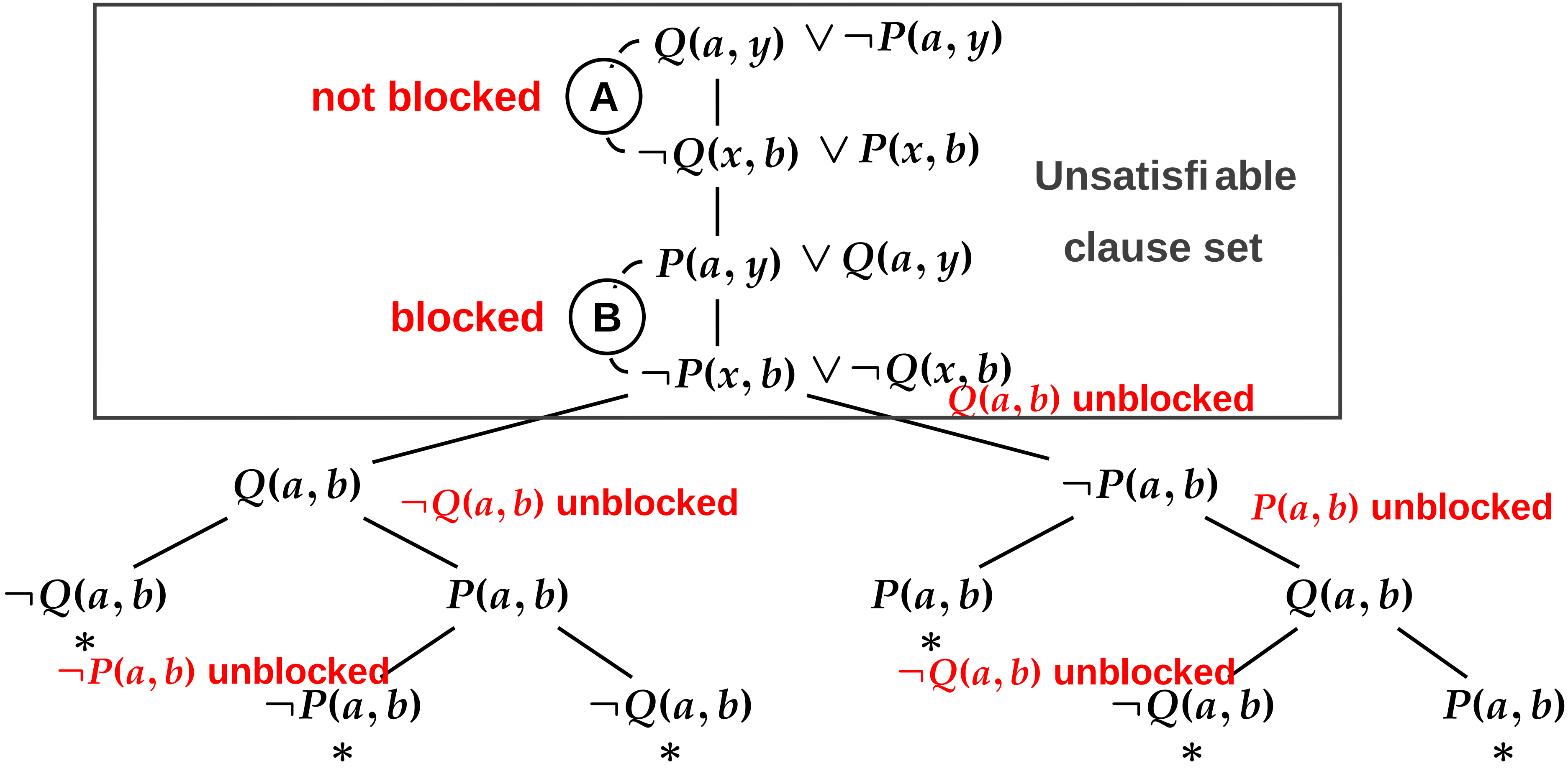

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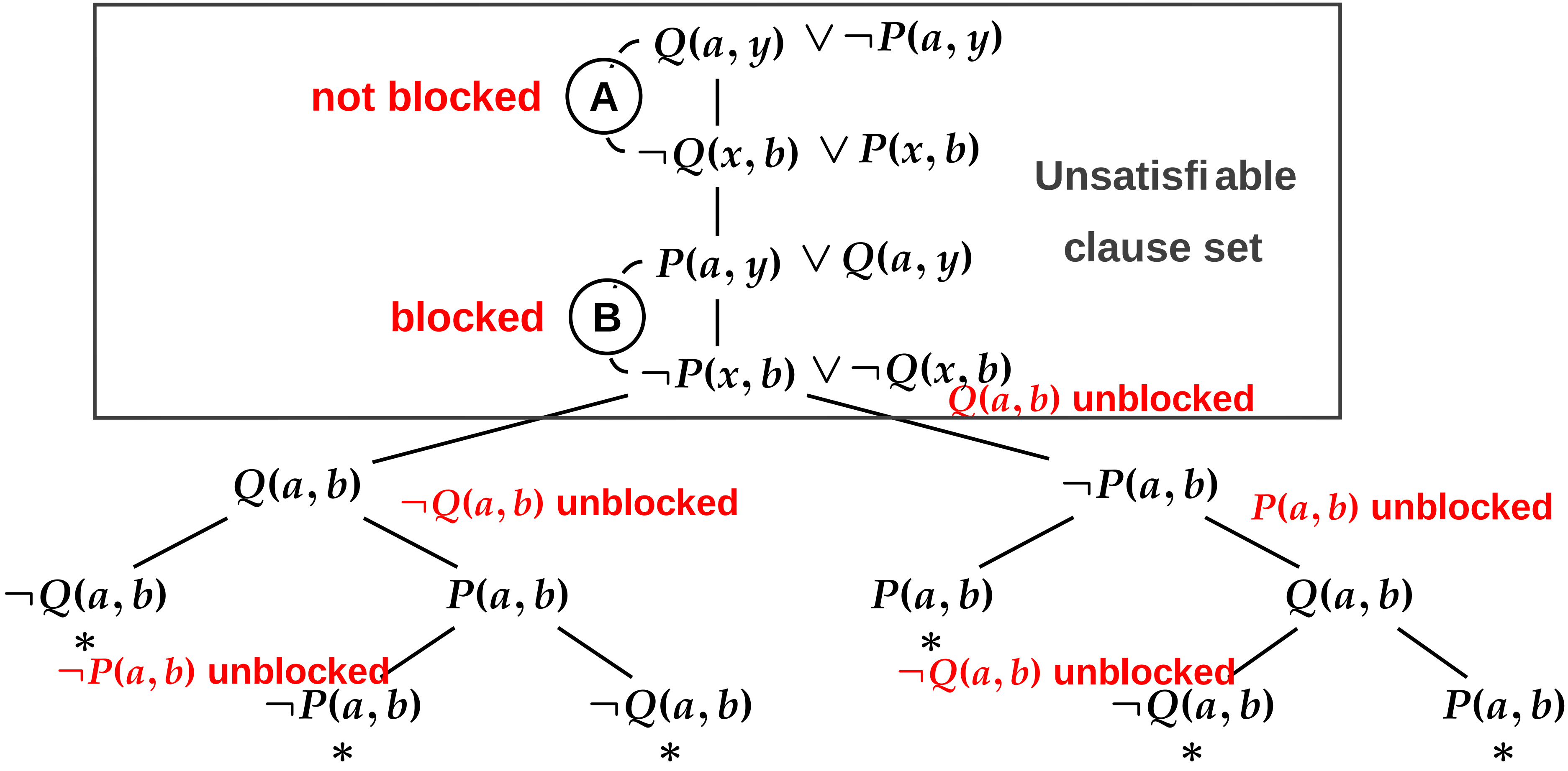

- Allowing link A to be applied, we initiate a series of blockings and unblockings that allow to refute the formula

The Basic Idea behind Completeness

- Completeness approach as in classical disconnection calculus:

saturated open tableau branch B^+

$$\Longrightarrow$$

consistent path P^* through Herbrand set

- P^* path literal in each ground clause is determined by CM-matcher
- Tricky part: There exists a matched literal in each ground clause
- Partial order of CM dynamically evolving with the branch
- Acyclicity of blocking relation ensures that partial order exists

FDPLL/ME vs. DCTP - Conceptual Difference

FDPLL/ME and DCTP use propositional version of current branch to determine branch closure. But:

DCTP

- **Branch is closed if it contains both $L\bot$ and $\overline{L}\bot$ (two clauses involved)**
- **Inference rule guided syntactically: find connection among branch literals**
- **n-way branching on literals of clause instance $L_1 \vee \cdots \vee L_n$**
  Can simulate FDPLL/ME binary branching to some degree (folding up)
- **Need to keep clause instances along current branch**

FDPLL/ME

- **Branch is closed if $\$$-version falsifies some single clause**
- **Inference rule guided semantically: find falsified clause instance**
- **Binary branching on literals L - $\overline{L}$ taken from falsified clause instance**
  Can simulate n-way branching clause literals in ground case
- **Need not keep *any* clause instance, but better cache certain subclauses (remainders) to support heuristics**

Theory Reasoning and Equality

Theory Reasoning (I)

Problem: Given a theory T and a clause set S. Is S T-unsatisfiable?

Verification applications: T is usually a combination of theories

(arithmetic, arrays, records, ...)

Example: Precondition: $x > 0$

Program: $y := x + 1$

Postcondition: $y > 1$

T is linear integer arithmetic. Show T-validity of

$$\forall x, y\ ((x > 0) \wedge (y = x + 1) \longrightarrow (y > 0))$$

More generally, have to show T-validity of a formula $\forall x\ \phi(x)$

Theory Reasoning (II)

Popular approch to prove T-validity of $\forall x\ \phi(x)$

- **Treat $\phi(x)$ as propositional formula**
- **Use DPLL (BDD, Tableaux, . . .) to get model $\{L_1, \ldots, L_n\}$ of $\phi(x)$**
- **Verify that $\forall(L_1 \wedge \cdots \wedge L_n)$ is T-valid (i.e. L_i's are interpreted again)**
- **The latter can be done for many useful theories (arrays, restricted arithmetic, integers, lists) and also combinations**
- **Bag of techniques to make this approach efficient**

Theory Reasoning (III)

Notation: $\forall x\ \phi(x)$ **is** T**-valid:** $\models_T \forall x\ \phi(x)$

General problem: show T**-validity under assumptions** Γ**:**

$$\Gamma \models_T \forall x\ \phi(x) \qquad\qquad (\Gamma \text{ could be } \forall x\ \psi(x))$$

Example (T **theory of equality, variables universally quantified):**

$$\{f(h(x)) \approx c,\ h(x) \approx x\} \models_T f(a) \not\approx c$$

Propositional reasoning is not enough:

$$\{f(h(\bot)) \approx c,\ h(\bot) \approx \bot\} \not\models_T f(a) \not\approx c$$

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How to discover required instances $f(h(a)) \approx c$ **and** $h(a) \approx a$ **?**
Propositional reasoning doesn't provide guidance!

Theory Reasoning (IV)

Dilemma:

- **Could enumerate ground instances or make heuristic choice**

  (current practice in verification tools, e.g. CVC Lite)

  - Inefficient, incomplete

  + Can use existing decision procedures for T

- **Use theory reasoner to compute T-unifiers**

  + Possibly complete and efficient, depending from T

  (see below for Inst-Gen with equality)

  - Does not exploit existing decision procedures for T,

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  (see below for Inst-Gen with equality)
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Perhaps the most pressing research problem!

Theory Reasoning for Equality

- Equality is by far the most important and mostly used theory
- Unlike other theories handled on the first-order level
- Different ways of integrating equality into instance based methods

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- **Different ways of integrating equality into instance based methods**
- **The easiest form: axiomatic equality handling**
- **Other methods all based on paramodulation:**
  - **Superposition-like [Bachmair and Ganzinger, 1994] eq-linking (disconnection calculus)**
  - **Disagreement linking (disconnection calculus)**
  - **Unit paramodulation and non-proper demodulation (Inst-Gen)**

Axiomatic Equality Handling

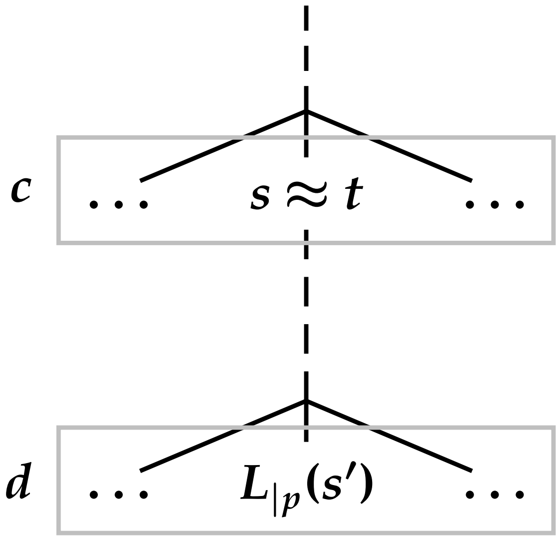
- Simplest form of treating equational problems
- No special inference rules or adaption of calculus/prover required
- Equality axioms added to input clause set
- Axioms for reflexivity, transitivity and symmetry
- Substitution axioms for all functors and predicate symbols. For example:

  $$x \approx y \rightarrow f(\ldots, x, \ldots) \approx f(\ldots, y, \ldots)$$

  for every argument position of every functor f
- Inefficient due to redundancy and incompatibility with orderings
- “Disconnects” altered terms from their clauses

Eq-Linking

- **Additional inference rule: tableau equivalent of paramodulation**



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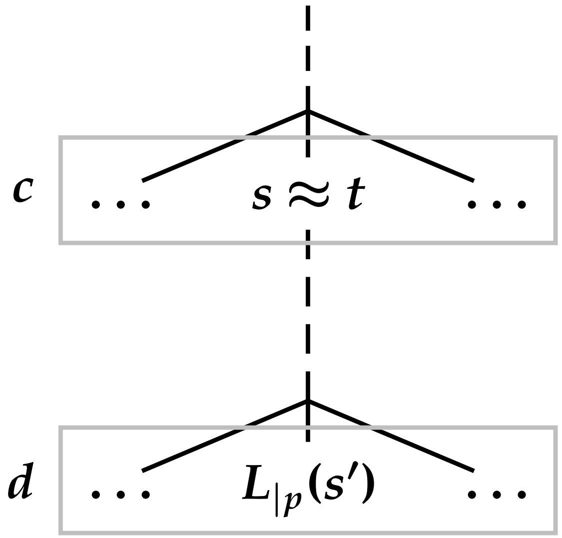

eq-link on path: one side s of equation and subterm s' unifiable with unifier σ

Eq-Linking

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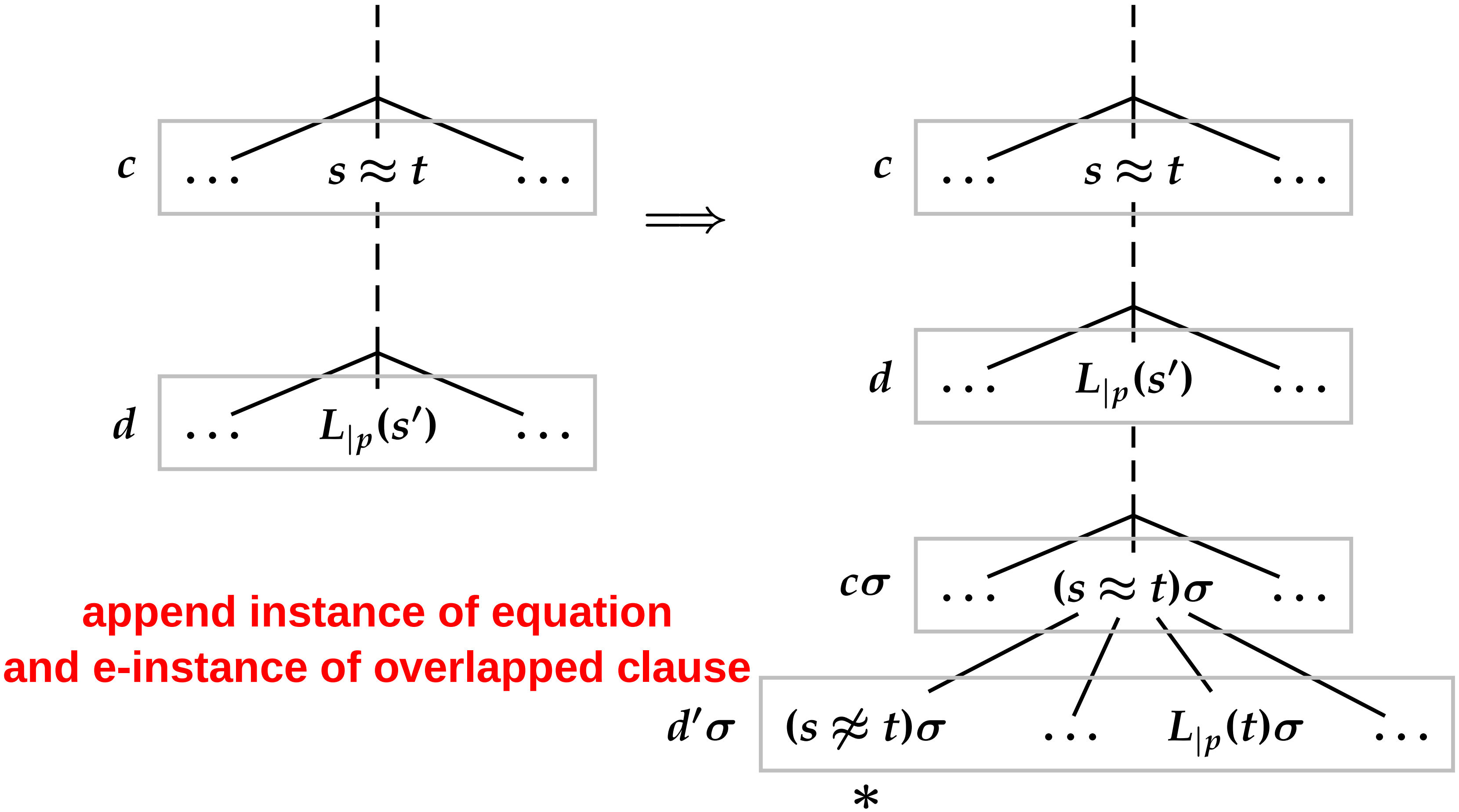

- **Negation of applied equation added to modified clause: *e-instance***

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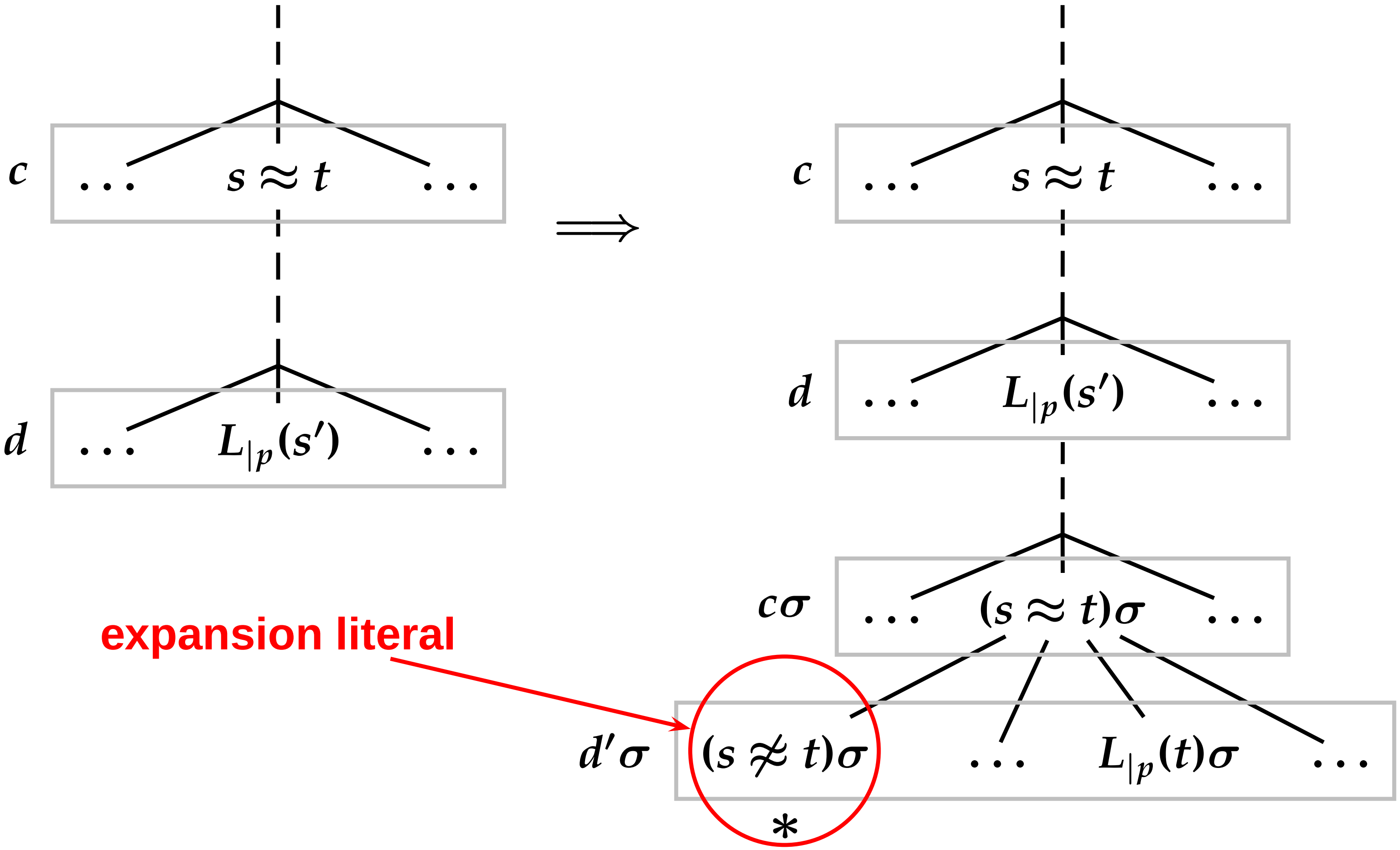

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Eq-Linking (II)

- Overlapping equation and overlapped literal form *eq-link*
- Expansion literals not necessary when eq-linking with unit equations
- Reflexivity linking rule required for completeness:

$$\frac{C \vee s \not\approx t}{C\sigma} \text{ where } \sigma \text{ is the most general unifier of } s \text{ and } t$$

- Unrestricted application of eq-linking introduces large amount of redundancy
- But: eq-linking also compatible with *term orderings*
- Ordered eq-linking allows destructive rewriting of subgoals

An Example Proof with Eq-Linking

Input

clauses

$$h(a) \not\approx c$$
$$h(e) \approx d$$
$$h(f) \approx h(b)$$
$$a \approx b$$
$$c \approx d$$
$$e \approx f$$

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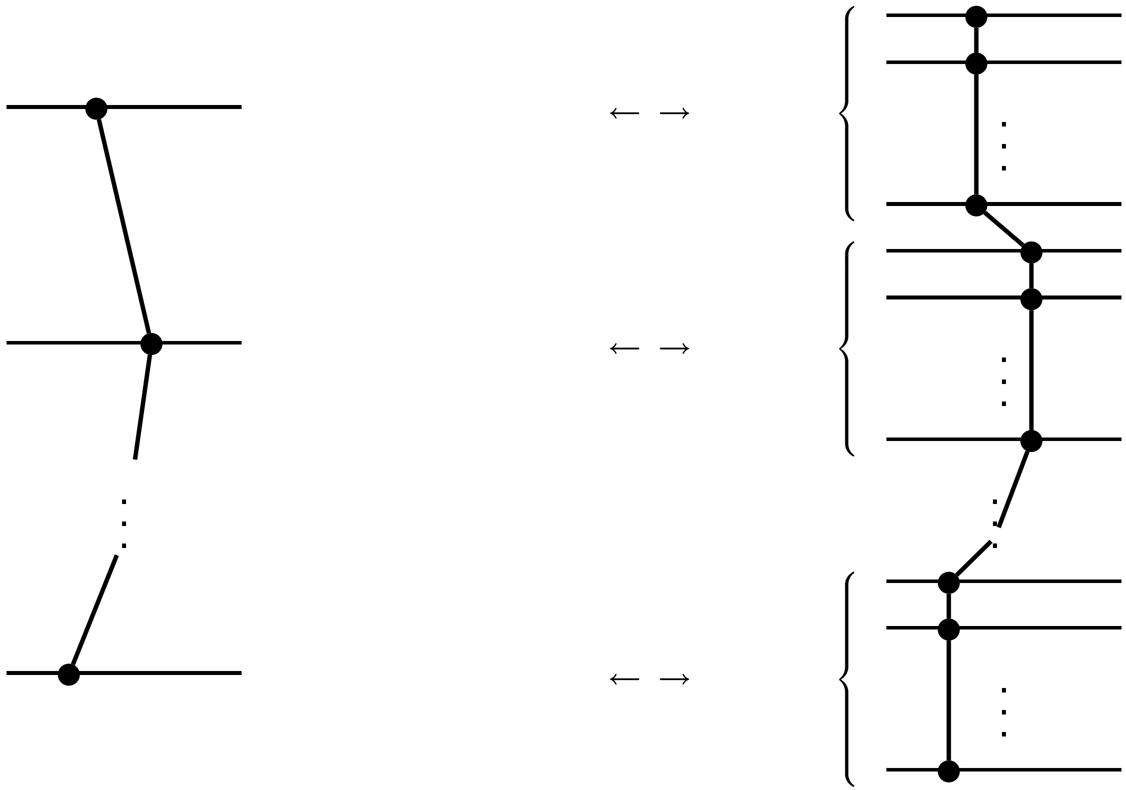
non-ground open branch (non-rigid) **ground Herbrand set**



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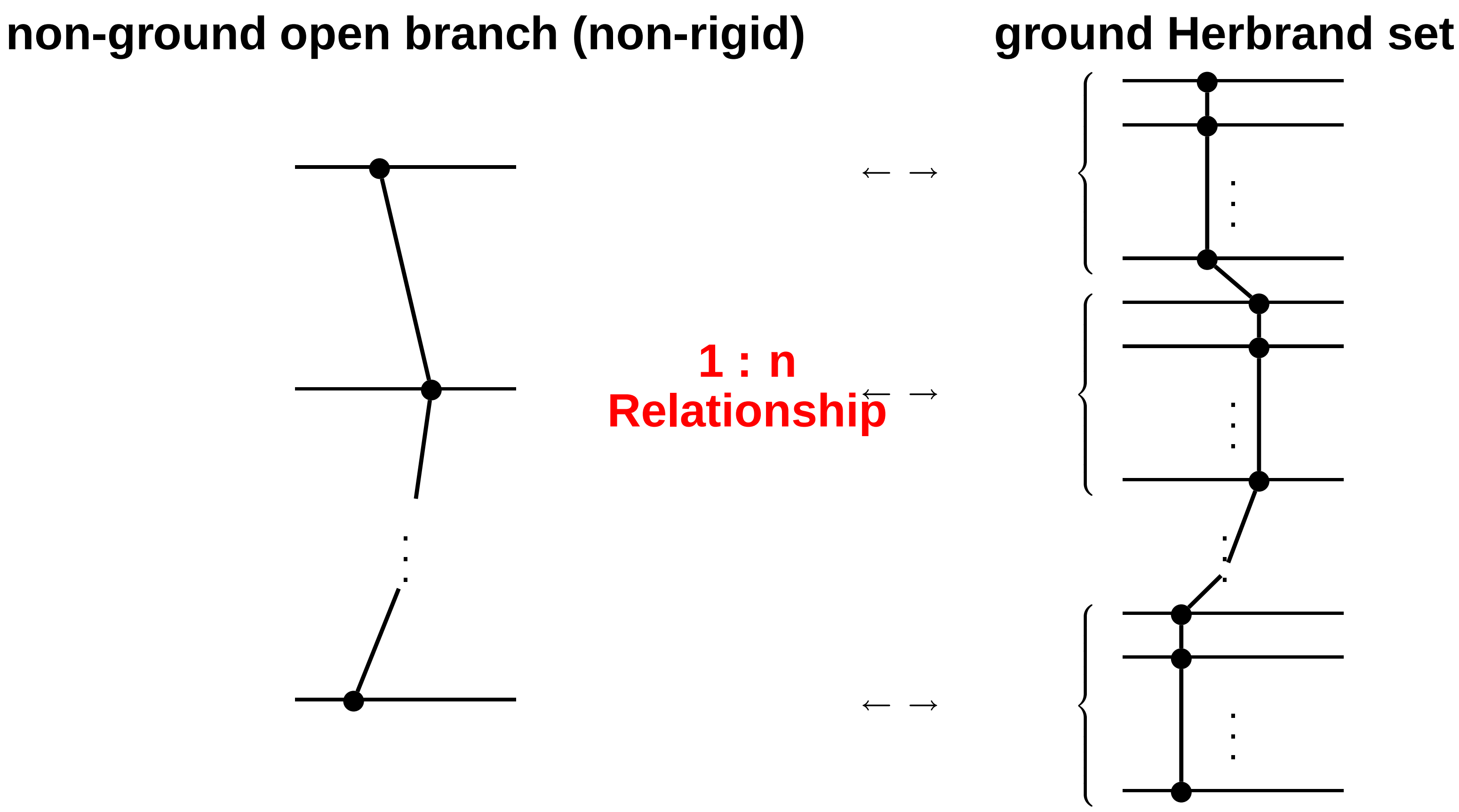

Completeness and Equality (II)

- **Now: one ground clause may correspond to many branch e-variants**

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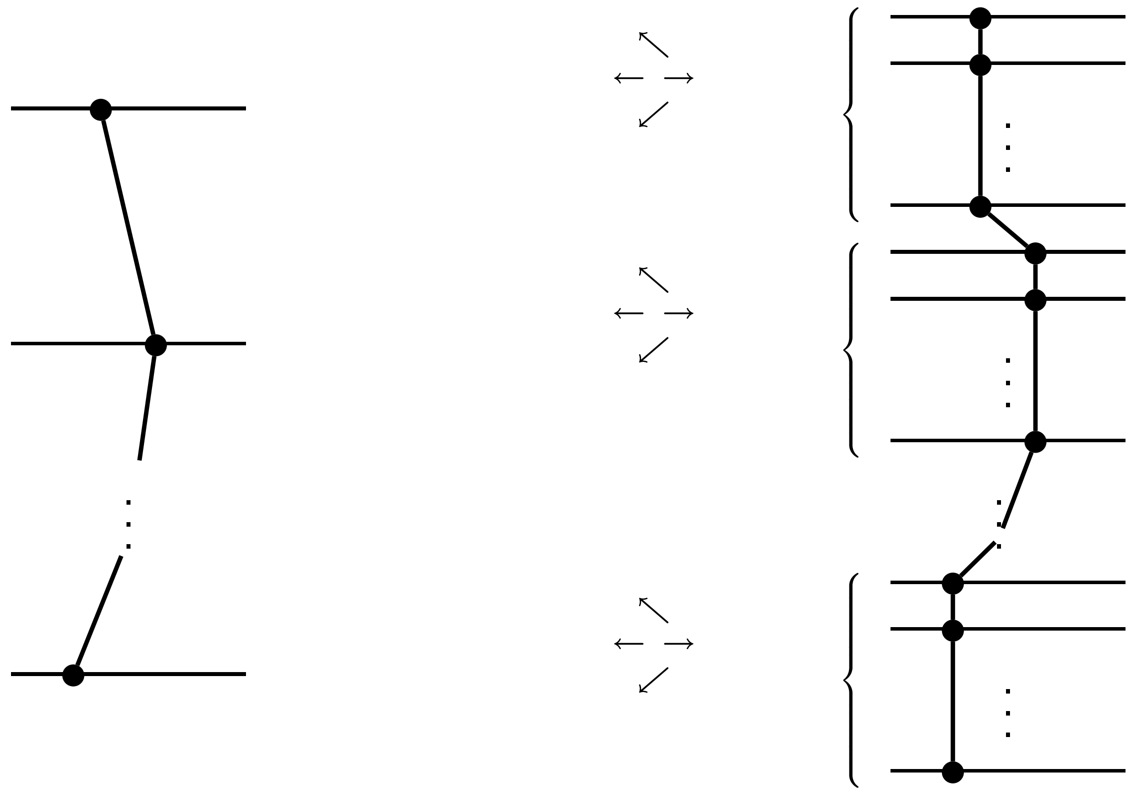

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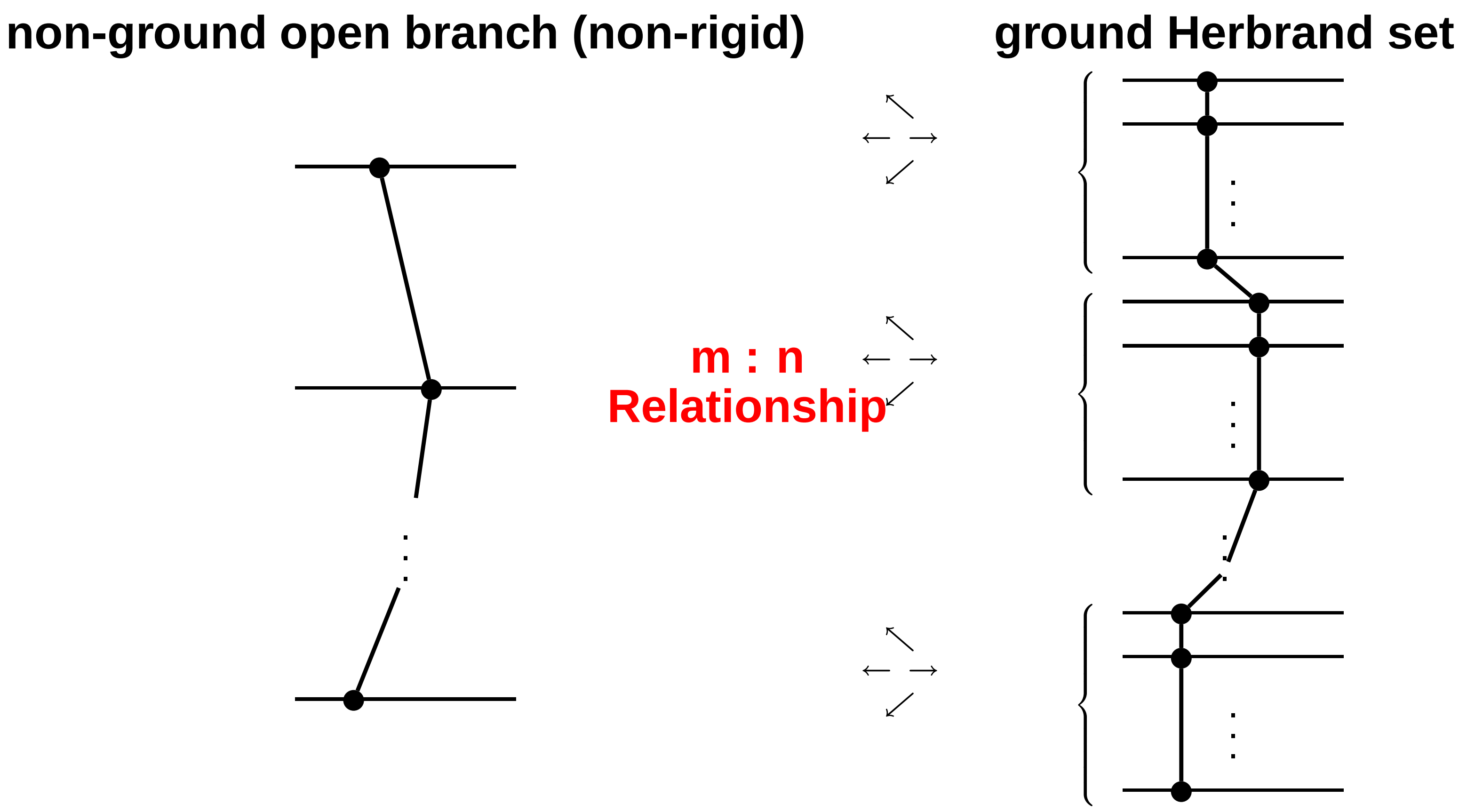

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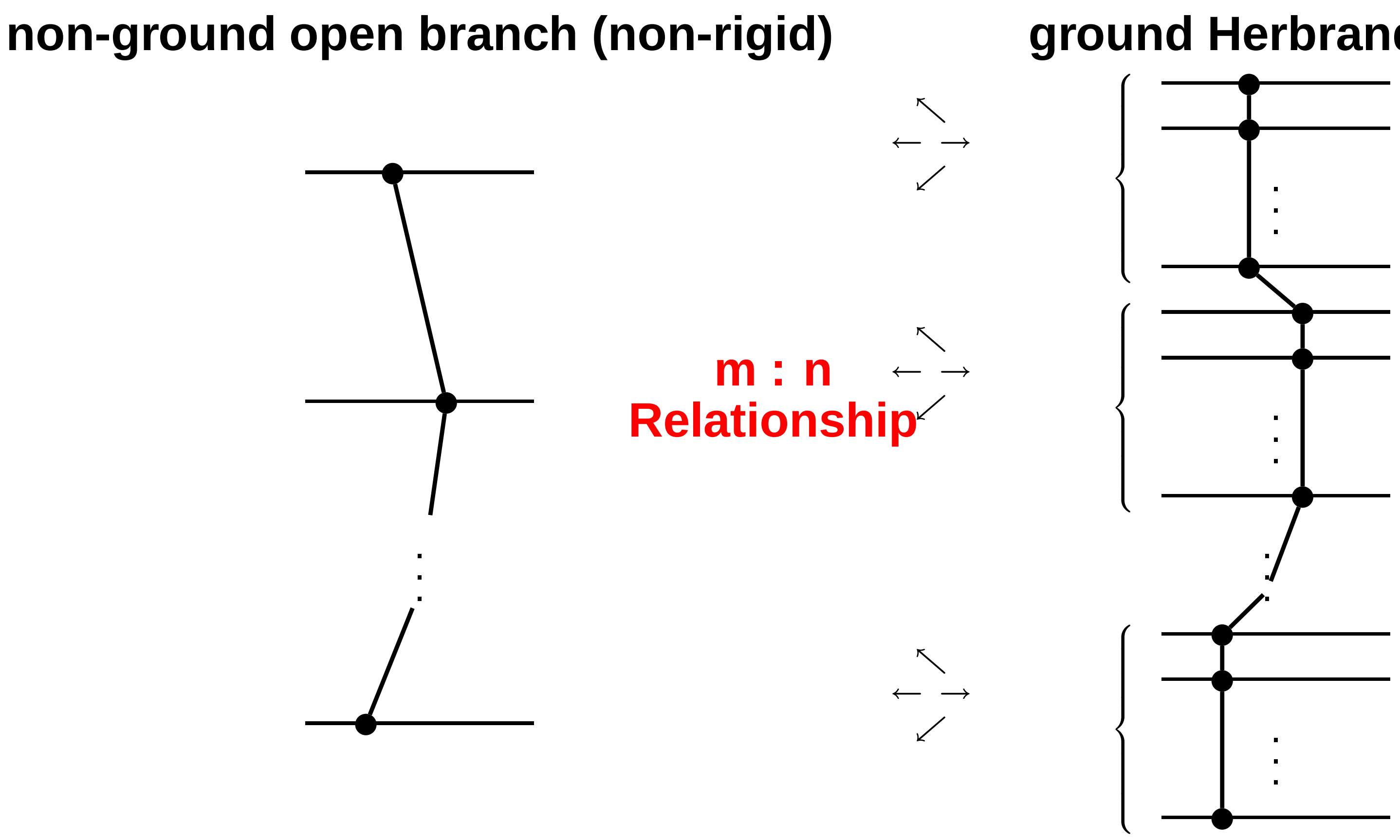

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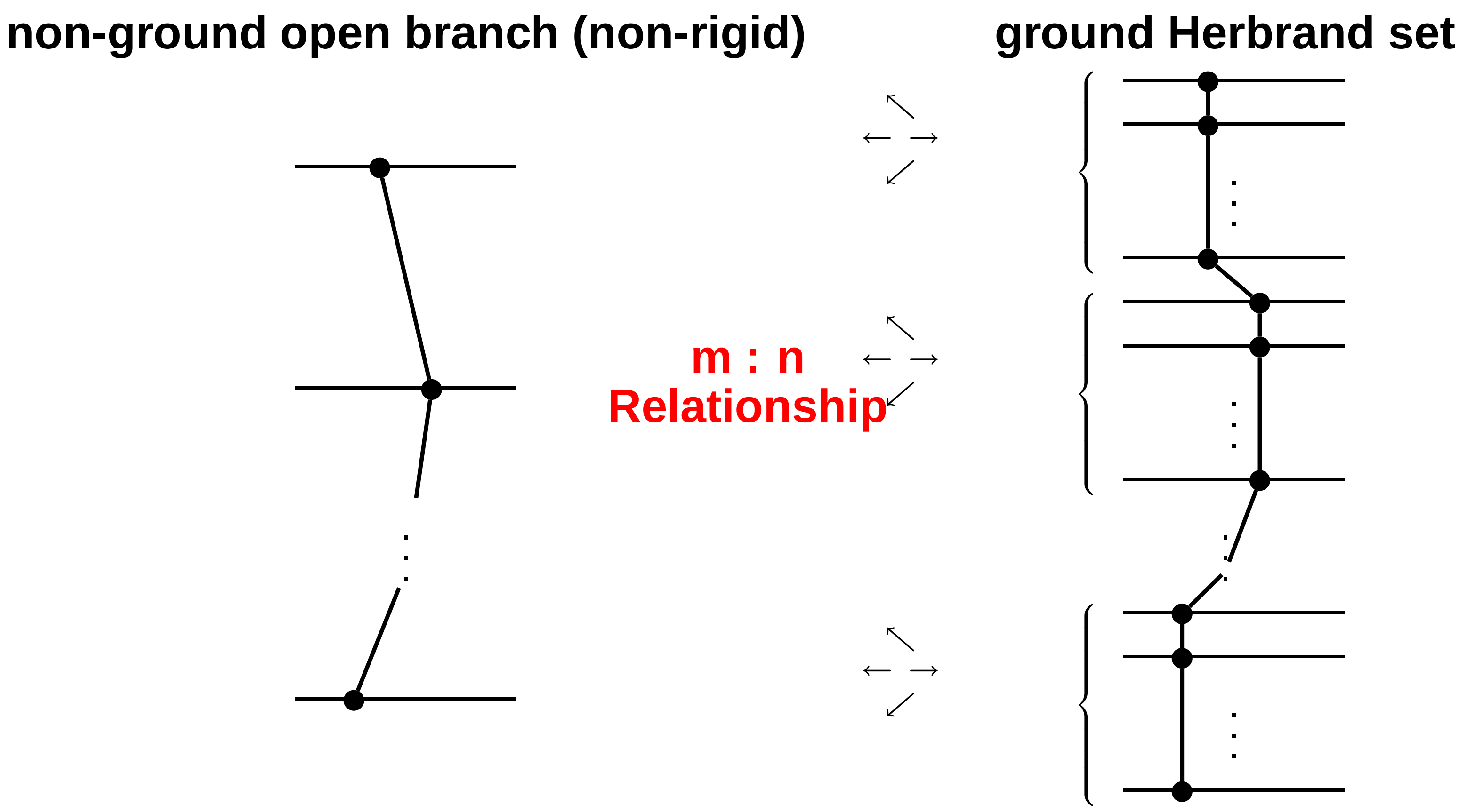


- Branch may pass through different literals in each of these e-variants
- One representative for each set of e-variants needs to be selected

Eager Variable Elimination

- Given: clause c with literal $l = x \not\approx t$ (x does not occur in t)
- l is a condition for the rest of the clause: $x \approx t \rightarrow c \setminus \{l\}$
- *Eager variable elimination* as a deterministic inference rule:

$$\frac{x \not\approx t \vee k_1 \vee \ldots \vee k_n}{k_1 \vee \ldots \vee k_n\{x/t\}}$$

- Helps keeping clause sizes down
- Care must be taken when eq-linking with unit equations
- Preservation of completeness is still an open problem

  [Gallier and Snyder, 1989]

Disagreement Linking

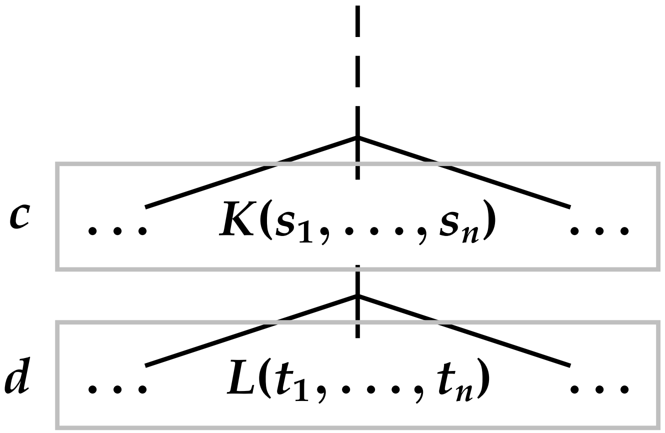
- Inspired by RUE-resolution [Digricoli and Harrison, 1986] and lazy paramodulation [Gallier and Snyder, 1989]
- Similar in behaviour to Brand- and STE-modification on the fly
- Based on the concept of *disagreement sets*:

  $L(s_1, \ldots, s_n)$ and $L(t_1, \ldots, t_n)$, $n \geq 0$ terms or literals

  Disagreement set: $\{s_1 \not\approx t_1, \ldots, s_n \not\approx t_n\}$
- Top-level unification of variable terms: disagreement substitution
- Eager variable elimination performed on disagreement set

Disagreement Linking (II)

- **Inference rule:**



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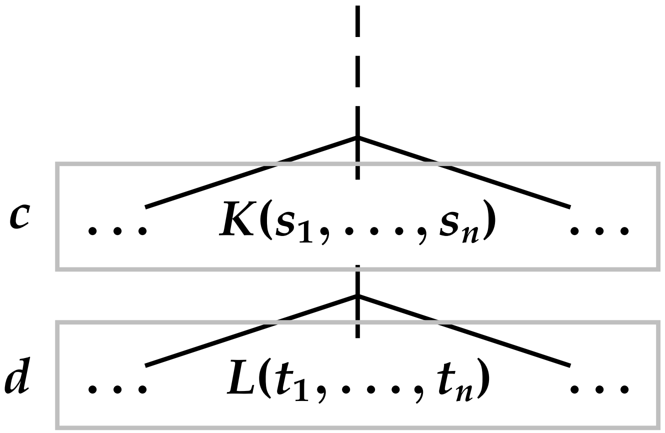

L and K share the same predicate symbol but have complementary signs

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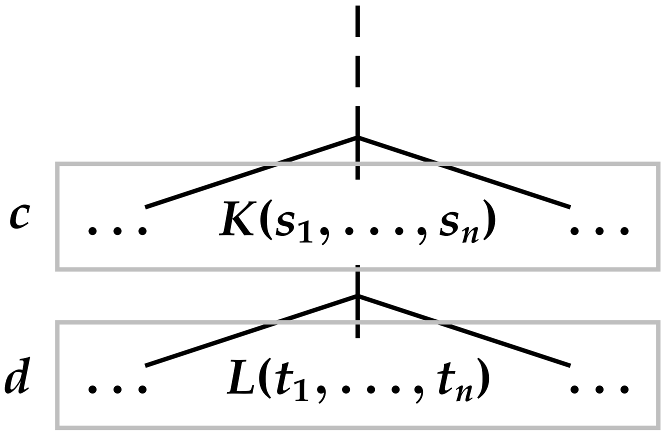

Some of the s_i and t_j are variables
forming the
disagreement substitution σ

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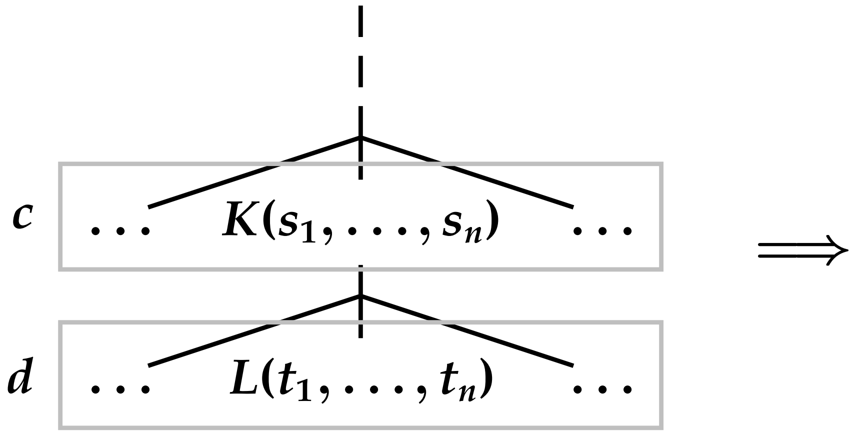

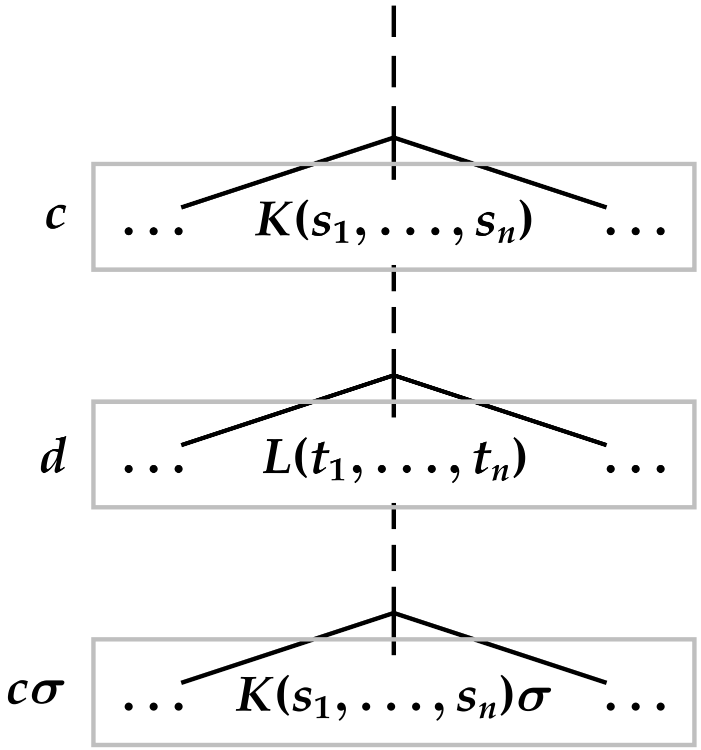

Expand tableau first
by disagreement instance $c\sigma$

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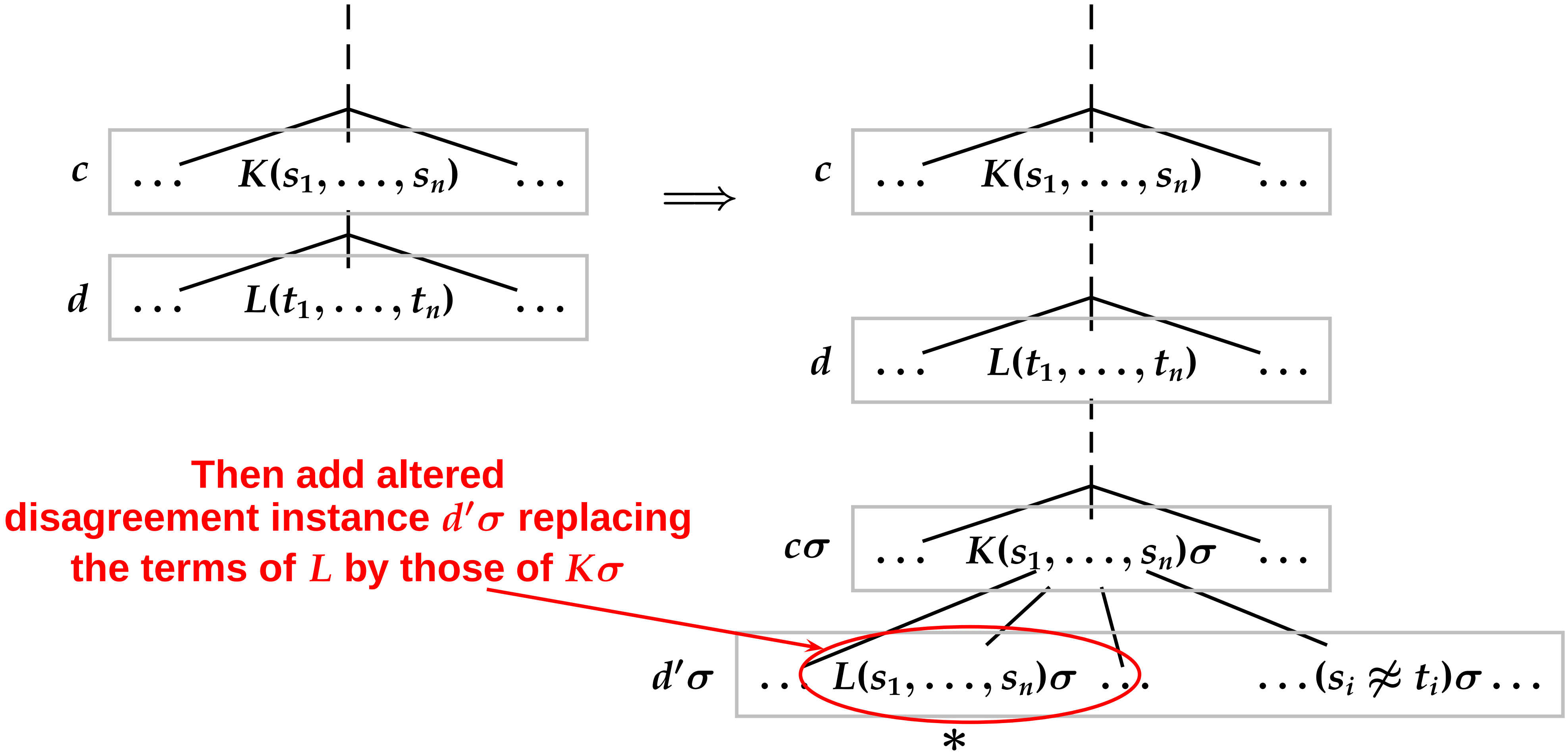

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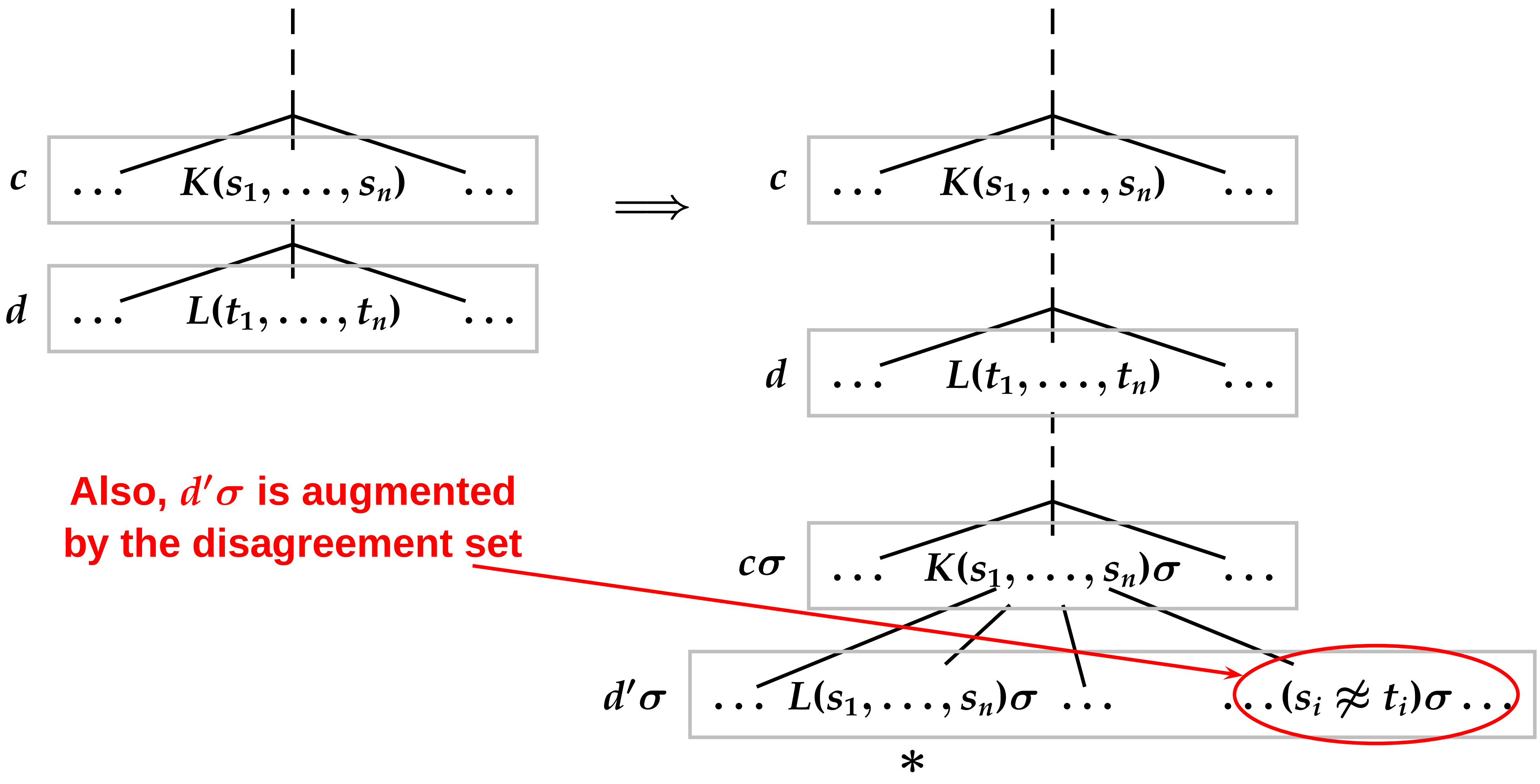

Disagreement Linking (III)

- **Decomposition rule required for completeness:**

$$\frac{f(s_1, \ldots, s_n) \not\approx f(t_1, \ldots, t_n)}{s_1 \not\approx t_1 \vee \ldots \vee s_n \not\approx t_n}$$

- **Also, Imitation rule for disequations of the form $x \neq f(x)$**
- **Incompatible with term orderings**
- **Disagreement linking cannot simulate full unification**
- **Additional standard linking necessary for instantiating terms**
- **Explicit symmetry handling required**
- **Sometimes improved recognition of e-satisfiability**

Inst-Gen and Equational Reasoning

- Recently developed equational reasoning for Inst-Gen [Ganzinger and Korovin, 2004]
- New method maintains separation of instance generation and ground satisfiability checking
- Instance generation not by linking, but by paramodulation rules
- Paramodulation performed on selected units
- Sound and complete
- Various techniques of redundancy elimination available

Inst-Gen and Equational Reasoning (II)

f.o. clauses

S

Inst-Gen and Equational Reasoning (II)

$$\begin{array}{ccc} \boxed{\begin{array}{c}\text{f.o. clauses}\\ S\end{array}} & \xrightarrow{\perp : \bar{x} \longrightarrow \perp} & \boxed{\begin{array}{c}\text{ground clauses}\\ S\perp\end{array}} \end{array}$$

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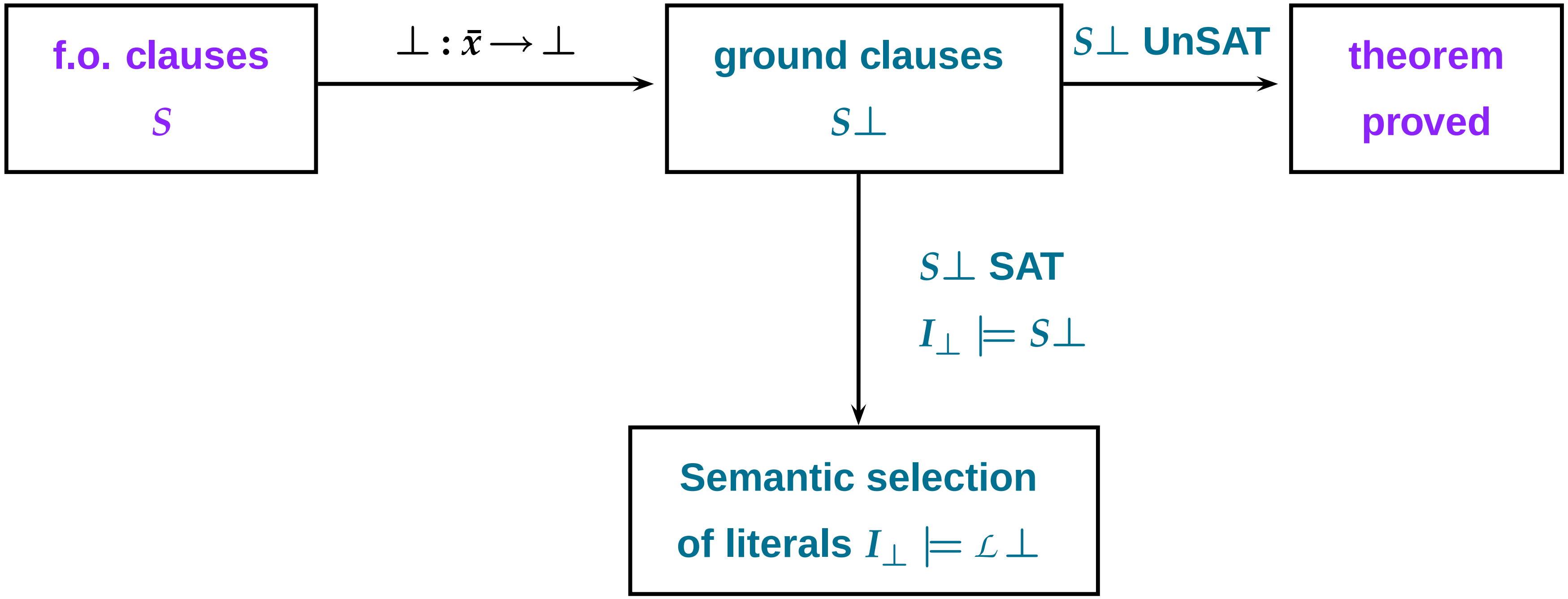
Inst-Gen and Equational Reasoning (II)



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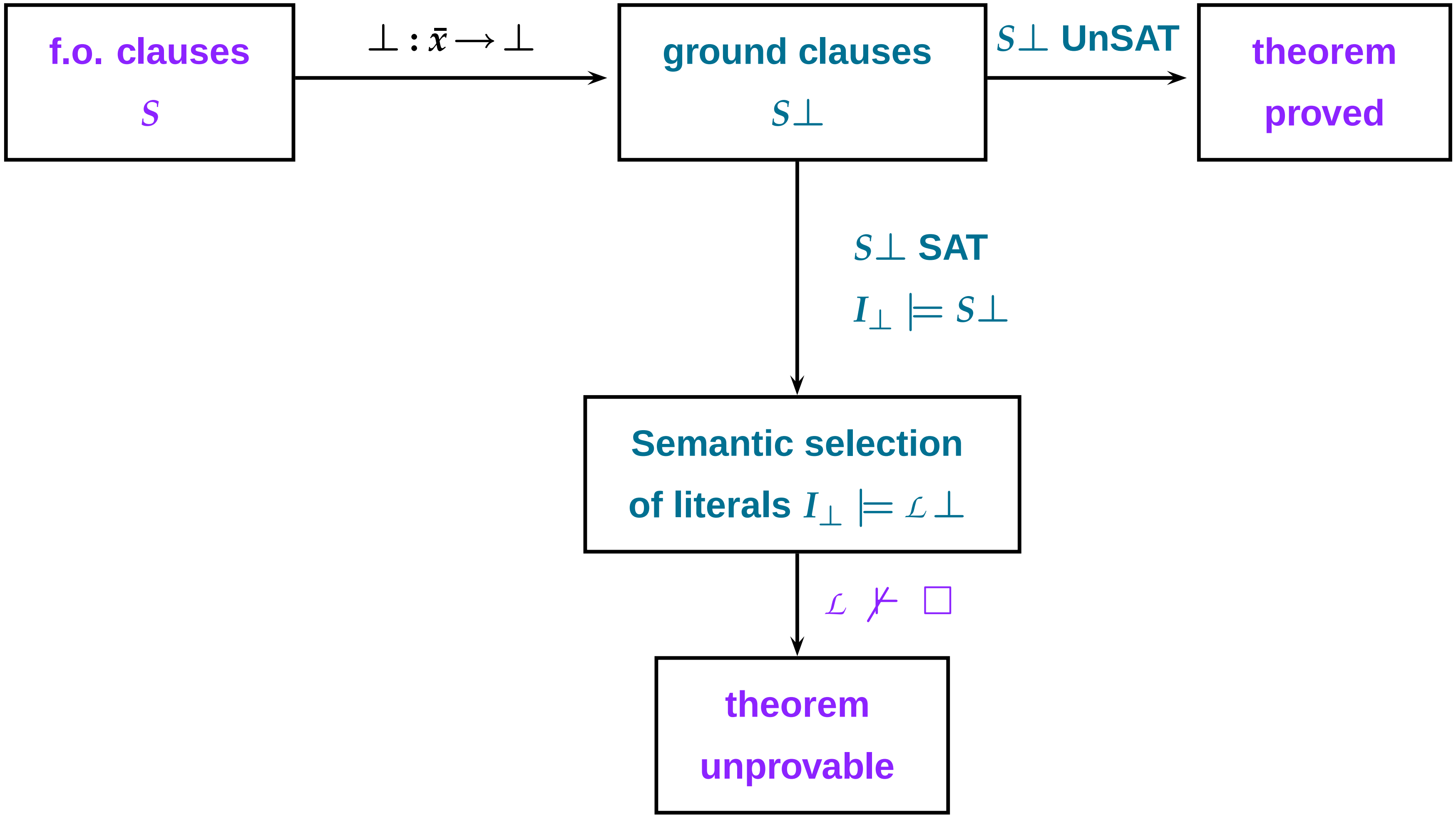

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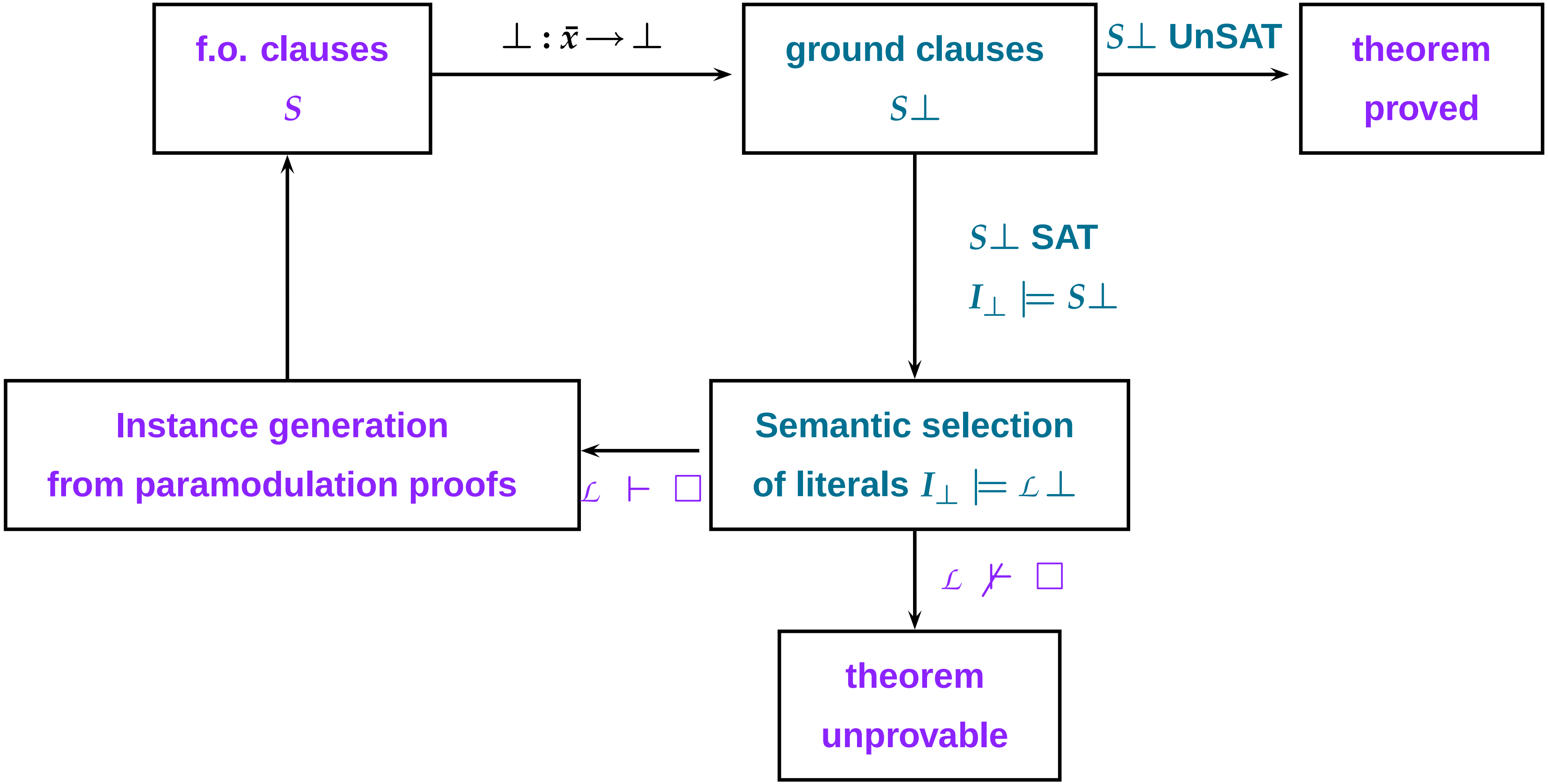

Assessment of Equality Handling Methods

- **Axiomatic Equality Handling**
  - **+ Can be used without modification of prover**
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  - - Incompatible with orderings, creates long clauses
- Inst-Gen Equational Instance Generation [Ganzinger and Korovin, 2004]
  - + Maintains separation of first-order and SAT part
  - + Good redundancy elimination, clauses do not grow in length
  - - Not implemented

Implementations and Techniques

Available Implementations

- **Some implementations of instantiation based methods have been realised**

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  - DARWIN: Model Evolution prover written in OCaml
  - DCTP: disconnection calculus tableau prover written in Scheme
- Of the implementations named above, DARWIN and DCTP participated in CASC-J2 (and CASC-20).
- Unfortunately, no implementation is available yet for Inst-Gen

Model Evolution - Darwin's Proof Procedure (I)

```
function darwin S
  // input: a clause set S
  // output: either "unsatisfiable"
  //          or a set of literals encoding a model of S
  let Context = ∅    // set of literals
  let L = ¬v  // (pseudo) literal
                    // Context ∪ {L} is the current context
  let Candidates = set of assert literals consisting of the
                          unit clauses in S
  try me(S, Context, L, Candidates)
  catch CLOSED -> "unsatisfiable"
```

$Candidates$: the literals eligible for application of assert or of split

Model Evolution - Darwin's Proof Procedure (II)

function $\mathit{me}(S, \mathit{Context}, K, \mathit{Candidates})$
  let $\mathit{Candidates}' =$
      $\mathit{add\_new\_candidates}(S, \mathit{Context}, K, \mathit{Candidates})$
  let $S' = S$ simplified by ***Subsume*** and ***Resolve***
  let $\mathit{Context}' = \mathit{Context} \cup \{K\}$ simplified by ***Compact***
  if $\mathit{Candidates}' = \emptyset$ **then** $\mathit{Context}'$ // Got a model of S'
  else
    let $L = \mathit{select\_best}(\mathit{Candidates}', \mathit{Context}')$
    if L is an assert literal **then**
      $\mathit{me}(S', \mathit{Context}', L, \mathit{Candidates}' \setminus \{L\})$ // assert L
    else
      try
        $\mathit{me}(S', \mathit{Context}', L, \mathit{Candidates}' \setminus \{L\})$ // left split on L
      catch CLOSED ->
        $\mathit{me}(S', \mathit{Context}', \overline{L}^{\mathrm{sko}}, \mathit{Candidates}' \setminus \{L\})$ // right split on L

Model Evolution - Darwin's Proof Procedure (III)

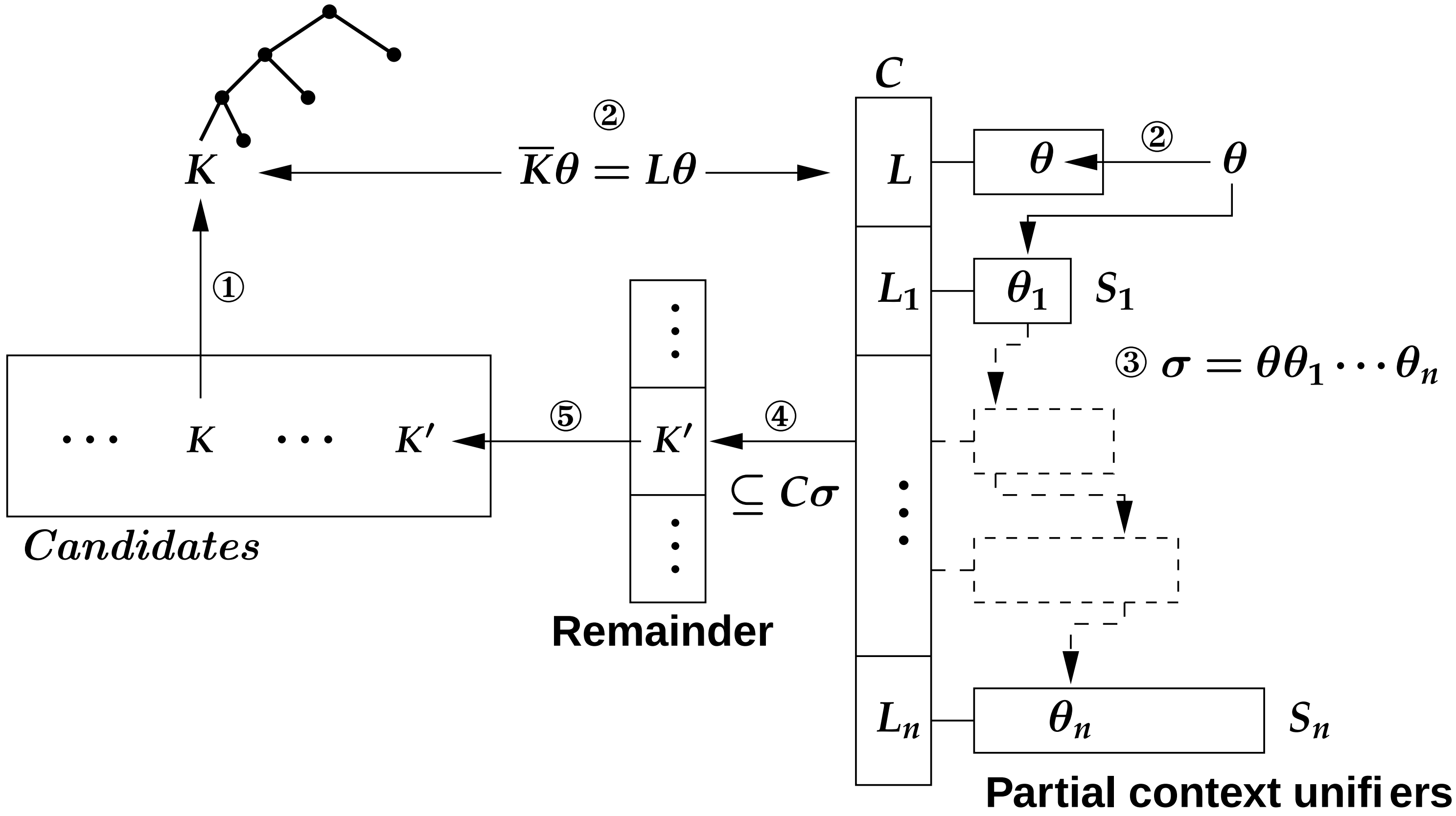
function ***add_new_candidates***(S, $Context$, L, $Candidates$)
  adds to $Candidates$ all assert literals from context unifiers involving L
  and one split literal from each remainder of a context unifier involving L
  raises the exception CLOSED if it finds a closing context unifier

Similar to semi-naïve evaluation of database rules (delta-iteration).

function ***select_best***($Candidates$, $Context$)
  returns the best assert or split literal in $Candidates$

To make select_best good and efficient, *all* theoretically required remainders are kept in store. See next slide.

Computing Remainders and Candidates [Baumgartner e



Partial context unifier: mgu of clause literal and context literal

① add literal K to context – ② compute all partial context unifiers θ of K and clause literals, and store with clause literals – ③ compute all context unifiers involving θ – ④ determine all remainders – ⑤ select K' from remainder and add to candidates (don't care nondeterminism)

Selection Heuristics for New Candidates

In decreasing preferrence:

1. **Universality** (**x - universally quantified; u - schematic variable**)

   $P(x)$ **is better than** $P(u)$

2. **Remainder Size**

   $P(a)$ **is better than** $P(b) \vee Q(b)$

3. **Term Weight**

   $P(a)$ **is better than** $P(f(a))$

4. **Generation**

   Prefer literals from remainders derived from elder context literals

   Rationale: prefer literals close to the original clause set

The Main Loop of DCTP

procedure disconnect($clauses$ **)**

select_initial_path;

create_links($initial\_path$ **);** $start\_sg$ **:= last(** $initial\_path$ **);**

solve_subgoal($start\_sg$, $links$, $initial\_path$ **);**

print("Proof");

The Main Loop of DCTP

```
procedure disconnect( clauses )
    select_initial_path;
    create_links( initial_path ); start_sg := last( initial_path );
    solve_subgoal( start_sg, links, initial_path );
    print( "Proof");

procedure solve_subgoal( sg, links, path )
    if ( ¬ forall_closed( sg ) ) then
      create_new_links( sg );
      if ( apply_linking_step( links ) ) then
        foreach new_sg ∈ ( new_subgoals )
                solve_subgoal( new_sg, links, initial_path ∪ sg );
        end
      else
        print( "Saturation state reached" ); stop;
      endif;
    endif;
```

Picking Up SAT Techniques

Merely a summary of what has been said before

Universal Variables and Unit Propagation

- **Picked up in OSHL, DCTP and ME with varying realization**

Lemma Generation (Learning in SAT)

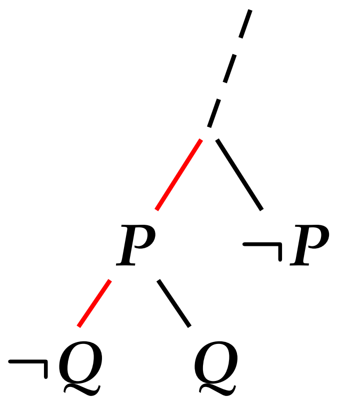
- **Local unit lemmas in DCTP**
- **Global lemma possible in ME (work in progress)**
  In DPLL: lemma clause determined from resolution derivation associated to closed subtree – idea lifts to ME

Other

- **Dependency directed backtracking (backjumping, tableau pruning): a must for any serious prover. . .**
- **DPLL splitting heuristics, randomized restarts – unexplored**

Unit Propagation in SAT

- **Unit propagation is a fundamental technique for efficient SAT proving**
- **Main technical motivation for Model Evolution calculus (see Part I)**



The current open branch
is indicated in red

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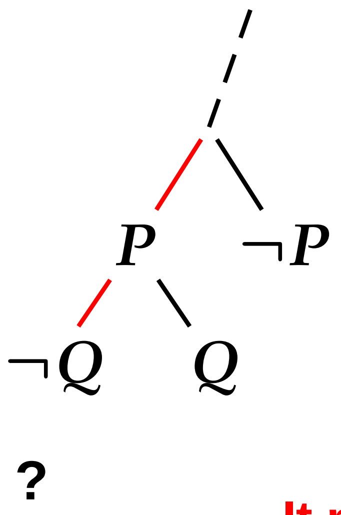

It must be decided over which variable to branch next

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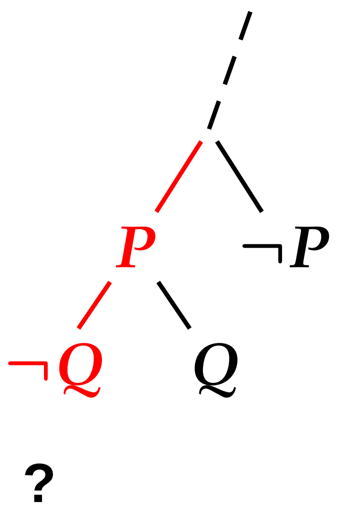

Input clauses

$\vdots$

$\neg P \vee Q \vee \neg R$

$\vdots$

The path context is used
to count down the length of input clauses

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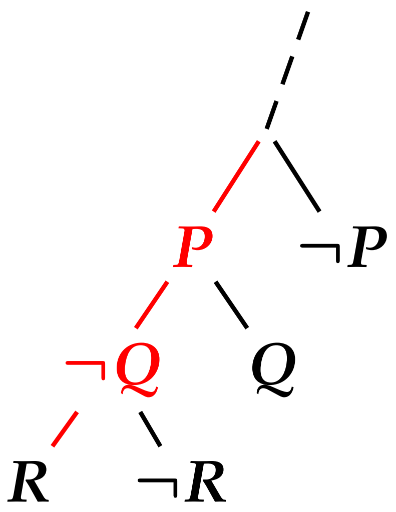

Input clauses

$$\vdots$$

$$\neg P \vee Q \vee \boxed{\neg R}$$

$$\vdots$$

In one input clause
only one unmatched literal remains
This literal is used for the next branching

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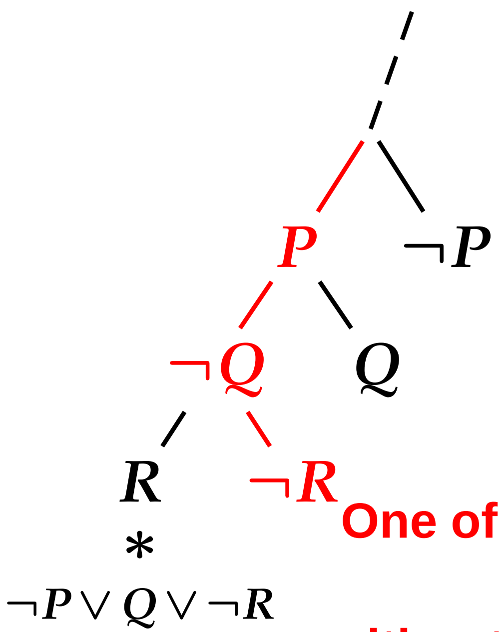

Input clauses

$$\vdots$$

$$\neg P \vee Q \vee \neg R$$

$$\vdots$$

One of the new branches can immediately be closed

The proof search continues effectively
without branching and with an extended path context

Unit Propagation in Disconnection Tableaux

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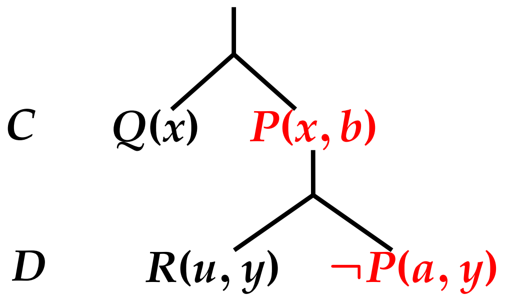

Link: potentially complementary literals on path

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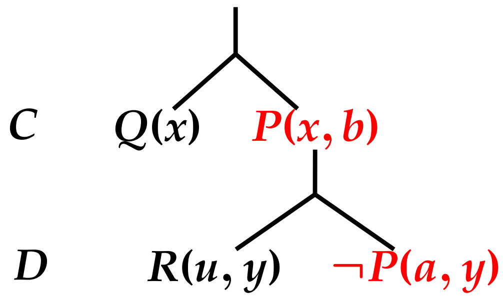
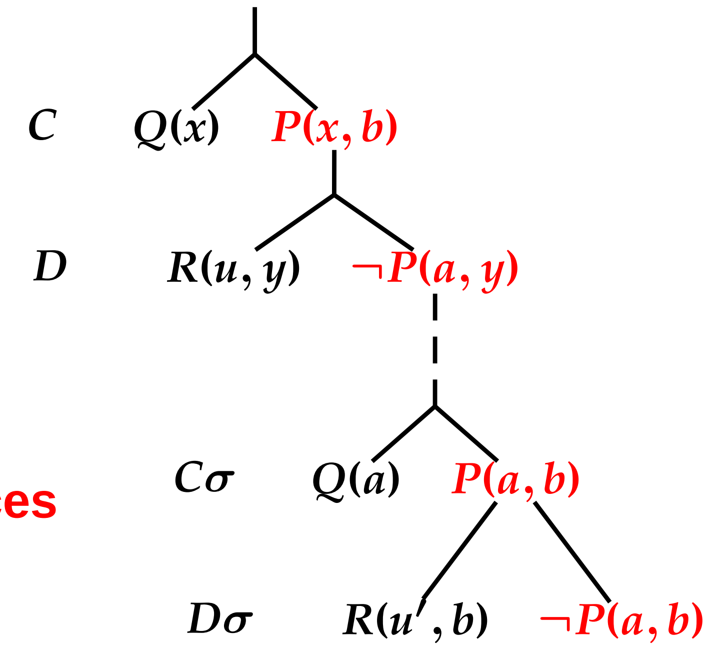


$\Rightarrow$



application of link creates linking instances

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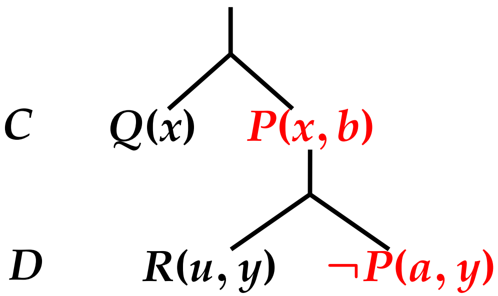

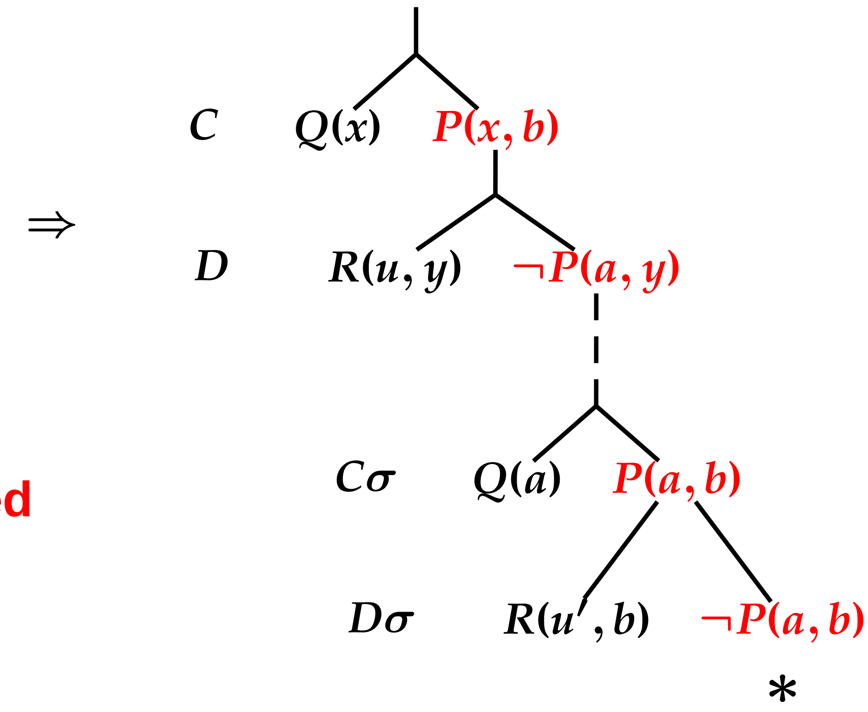

new open paths are created

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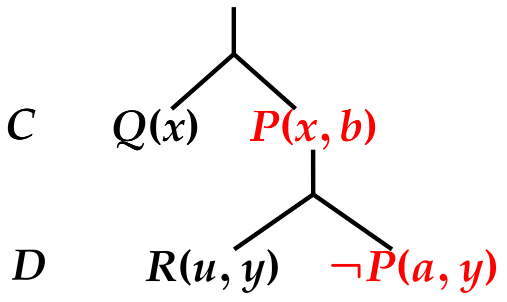

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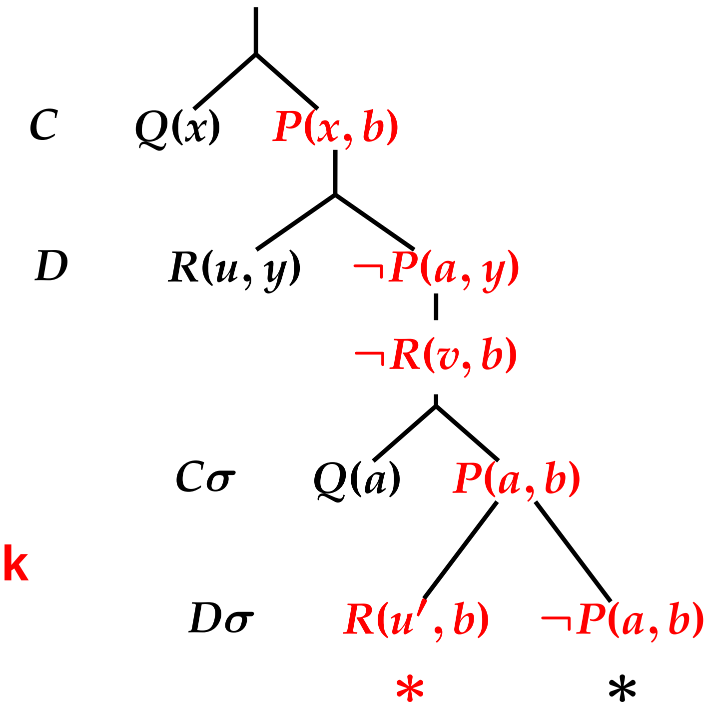

path context is used
to count down open paths created by link

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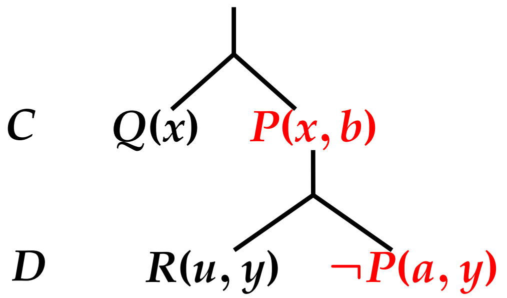

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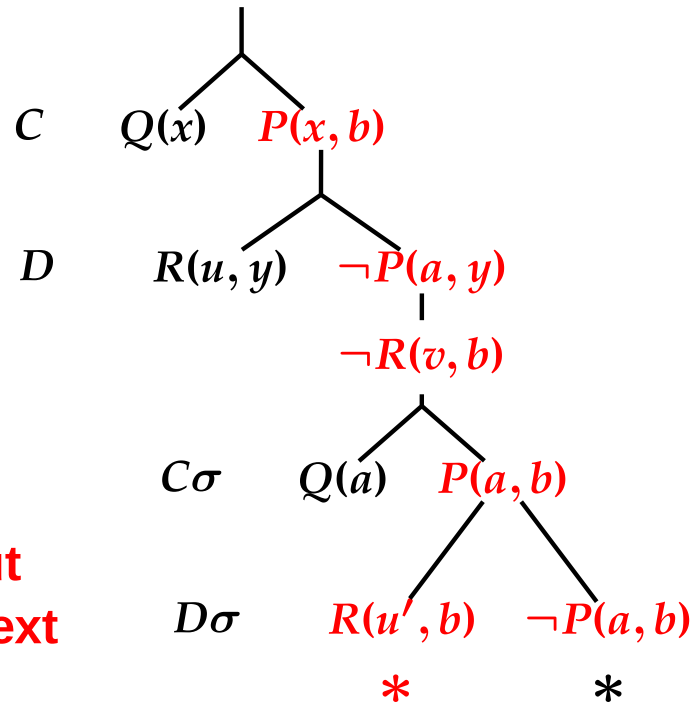

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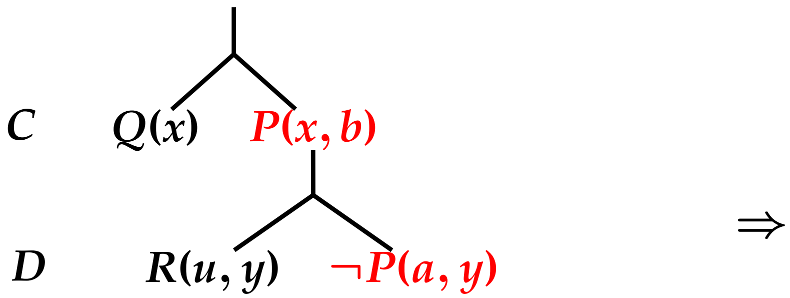

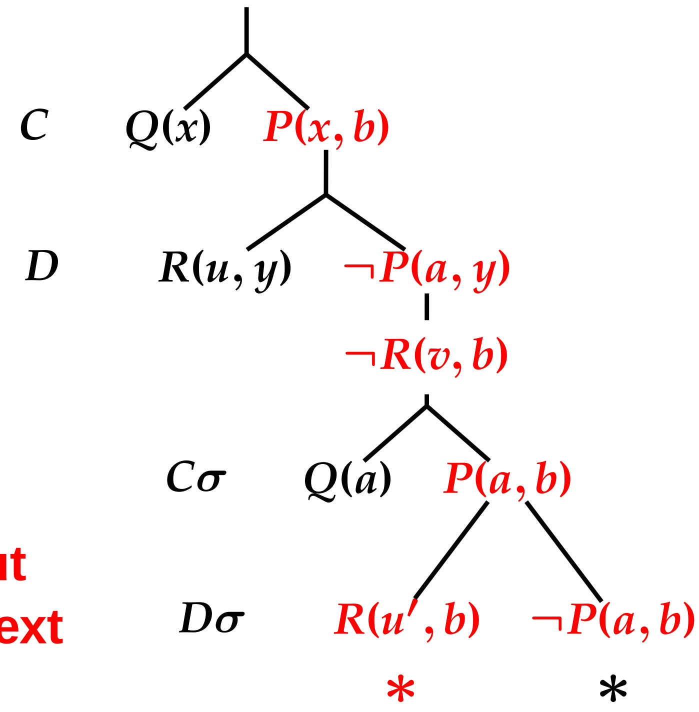

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- **Method need not terminate due to new links by new instances**
- **Selection heuristic instead of deciding strategy**

Conclusions

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- Two-level methods able to capitalise on successful SAT technology
- Single-level methods successful in their own right
- Some SAT techniques are liftable to first-order
- Possible topics for future research
  - Incorporating theory decision procedures
  - Deciding interesting classes of first-order logic
  - Comparing calculi (e.g. stepwise simulation or wrt. instance sets)
  - Improving implementations (more SAT techniques, heuristics, data structures)