Finite Quantification in Hierarchic Theorem Proving

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Overall Goal

Theorem Proving in Hierarchic Combinations of Specifications

Foreground Specification (FG)
Axioms: Lists, Arrays
Definitions: Length, isSorted

extends

Background Specification (BG)
- Linear integer arithmetic

$\overset{?}{\models}$

Conjecture

Main issue

Quantifiers: complete theorem proving is theoretically impossible
Problem: incompleteness: "no refutation" $\not\Rightarrow$ "countersatisfiable"

Calculi for Hierarchic Reasoning

SMT: DPLL(T) + instantiation heuristics (CVC4, Z3,...)

Model evolution with LIA constraints [B Tinelli 2008, 2011]

Sequent calculus [Rümmer 2008]

Theory instantiation [Korovin 2006]

LASCA [Korovin Voronkov 2007]

Hierarchic superposition

[Bachmair Ganzinger Waldmann 1994, Althaus Weidenbach Kruglov 2009, Weidenbach Kruglov 2012, B Waldmann 2013]

This work

Recover completeness for finitely quantified fragment

Can be used on top of hierarchic superposition and SMT

Hierarchic Specifications

Background (BG) specification consists of

Sorts, e.g. { **int** }

Operators, e.g. { **0, 1, -1, 2, -2, ..., -, +, >, ≥**} Parameters e.g. { **m, n, α** }

Models, e.g. linear integer arithmetic

Foreground (FG) specification extends BG specification by

New sorts, e.g. { **array** }

New operators, e.g.

{ **read: array × int ↦ int,**

write: array × int × int ↦ array,

a: array }

First-order clauses, e.g. array axiom

$\{ \mathsf{read}(\mathsf{write}(a, i, x), i) \approx x,$

$\mathsf{read}(\mathsf{write}(a, i, x), j) \approx \mathsf{read}(a, j) \vee i \approx j \}$

Finite saturation by superposition

Hierarchic Specifications

Array axioms from above

(1) $\text{read}(\text{write}(a, i, x), i) \approx x$

(2) $\text{read}(\text{write}(a, i, x), j) \approx \text{read}(a, j) \vee i \approx j$

Additional clauses

(3) $\text{read}(\text{a}, i) \leq \text{read}(\text{a}, j) \vee \neg(i < j) \vee i \notin [1..1000] \vee j \notin [1..1000]$

// Array **a** is sorted in the range [1..1000]

(4) $1 \leq \text{m} \wedge \text{m} < 1000$

(5) $\text{read}(\text{a}, \text{m}) < \text{read}(\text{a}, \text{m}+1)$

Can't we directly use superposition?

Contributions of this paper

A general method for model computation on top of HSP/SMT, e.g.

$\mathbf{a} \mapsto$	5	5	5	...	5	6	...	6	6	6
	1	2	3	...	50	51	...	998	999	1000
					m					

Hierarchic Specifications

Models of hierarchic specifications

Must satisfy the FG clauses, and
must leave the interpretation of the BG sorts and operators unchanged (*conservative extension*):

- distinct BG elements may not be identified (*no confusion*), and
- no new elements may be added to BG sorts (*no junk*)

Hierarchic superposition calculus (HSP)

Extension of the superposition calculus for hierarchic specifications

Calls BG-solver to decide BG-unsatisfiability of BG clauses

Complete under assumptions: **sufficient completeness**, compactness

The clause set (1)-(5) is not sufficiently complete

Finite saturation does not mean "satisfiable (wrt hierarchic interpretations)"

Sufficient Completeness

Sufficient Completeness

In every model of the FG clauses, every ground FG term that has a BG sort must be equal to some BG term

Example

(3) $\mathrm{read}(a, i) \leq \mathrm{read}(a, j) \vee \neg(i < j) \vee i \notin [1..1000] \vee j \notin [1..1000]$

(5) $\mathrm{read}(a, m) < \mathrm{read}(a, m{+}1)$

is not sufficiently complete, admits junk:

Domain: { 0, -1, 1, -2, 2, ..., NaN }

Interpret: $\mathrm{read}(a, i) \mapsto \mathrm{NaN}$ $(\mathrm{NaN} < \mathrm{NaN}) \mapsto \mathrm{true}$ $(\mathrm{NaN} \leq \mathrm{NaN}) \mapsto \mathrm{true}$

Consequence

Finite saturation of (1) - (5) under HSP does not mean anything

Next goal: recover sufficient completeness for **finitely quantified clauses**

Finitely Quantified Clauses

Definition

A clause C is *finitely quantified* if for every BG variable x occurring under a BG sorted FG operator, C contains a domain declaration of the form $x \notin [\mathrm{l..u}]$, where l and u are concrete integers.

Examples

$$(3) \quad \mathrm{read}(a, i) \leq \mathrm{read}(a, j) \vee \neg(i < j) \vee i \notin [1..1000] \vee j \notin [1..1000]$$

$$(5) \quad \mathrm{read}(a, m) < \mathrm{read}(a, m{+}1)$$

$$f(i{+}1, f(j, 2) + 1) > \boldsymbol{\alpha} + y \vee y > 0 \vee i \notin [1..1000] \vee j \notin [10..100]$$

(Rationale: using “large” domains is useful enough in practice)

Observation: only finitely many ground instances wrt BG sorted FG terms

Sufficient Completeness for Finitely Quantified Clauses

(3) $\text{read}(a, i) \leq \text{read}(a, j) \vee \neg(i < j) \vee i \notin [1..1000] \vee j \notin [1..1000]$

Alternative 1

Force mapping of relevant read-terms to integers by adding unit clauses

$\text{read}(a, 1) \approx 3$

$\text{read}(a, 2) \approx 5$

...

$\text{read}(a, 999) \approx 4$

$\text{read}(a, 1000) \approx 7$

Properties

Recovers sufficient completeness

Soundness and completeness by exhaustive search through mappings

Practically useless

Sufficient Completeness for Finitely Quantified Clauses

(3) $\mathrm{read}(a, i) \leq \mathrm{read}(a, j) \vee \neg(i < j) \vee i \notin [1..1000] \vee j \notin [1..1000]$

Alternative 2

Force mapping of relevant read-terms to integers by adding unit clauses

$\mathrm{read}(a, 1) \approx \alpha_1$

$\mathrm{read}(a, 2) \approx \alpha_2$

...

$\mathrm{read}(a, 999) \approx \alpha_{999}$

$\mathrm{read}(a, 1000) \approx \alpha_{1000}$

where α_i is a fresh parameter

Properties

Recovers sufficient completeness

Supplants outer loop by BG constraint satisfaction problem

Still practically useless

Sufficient Completeness for Finitely Quantified Clauses

(3) $\mathrm{read}(a, i) \leq \mathrm{read}(a, j) \vee \neg(i < j) \vee i \notin [1..1000] \vee j \notin [1..1000]$

Alternative 3 (taken)

Add unit clauses to express **default interpretation** with **exceptions**

$\mathrm{read}(a, i) \approx \alpha_0 \vee i \notin [1..1000] \setminus \{50, 60\}$

$\mathrm{read}(a, 50) \approx \alpha_{50}$

$\mathrm{read}(a, 60) \approx \alpha_{60}$

where α_i is a fresh parameter

Properties

Recovers sufficient completeness

Basis for procedure in paper

- Start with a default interpretation $\mathrm{read}(a, i) \approx \alpha_0 \vee i \notin [1..1000]$
- Modify by adding exceptions like 50, 60 in a conflict-driven way until model found or unsatisfiable

Next: idea of this method

Our Method - First Round

Given clause set $N[\Delta_x]$, where $\Delta_x = [1..1000]$

(1) $f(x) \not\approx x \lor x \notin [1..1000]$

(2) $f(5) \approx 8$

(3) $f(8) \approx 5$

Current set of exceptions $\Pi_x \subseteq \Delta_x$

Initially $\Pi_x = \{\}$

Finite Domain Transformation $M = FD(N[\Delta_x], \Pi_x)$

(f)	$f(x) \approx \alpha_0 \lor x \notin [1..1000]$	default interpretation for $f(x)$ in (1)
(1f)	$\alpha_0 \not\approx x \lor x \notin [1..1000]$	(f) applied to (1)
(2)	$f(5) \approx 8$	
(3)	$f(8) \approx 5$	

Now use HSP to check satisfiability

Our Method - First Round

Finite Domain Transformation $M = FD(N[\Delta_x], \Pi_x)$

(f)	$f(x) \approx \alpha_0 \vee x \notin [1..1000]$
(1f)	$\alpha_0 \not\approx x \vee x \notin [1..1000]$
(2)	$f(5) \approx 8$
(3)	$f(8) \approx 5$

M is unsatisfiable, take { $f(5) \approx \alpha_0$, $f(8) \approx \alpha_0$, (2), (3) }, HSP detects this

Maximal sub-domain $\Gamma_x = [1..7] \subseteq \Delta_x$ recovers satisfiability ($\alpha_0 \mapsto 8$)

(f)	$f(x) \approx \alpha_0 \vee x \notin [1..7]$
(1f)	$\alpha_0 \not\approx x \vee x \notin [1..7]$
(2)	$f(5) \approx 8$
(3)	$f(8) \approx 5$

Satisfiable

(f)	$f(x) \approx \alpha_0 \vee x \notin [1..8]$
(1f)	$\alpha_0 \not\approx x \vee x \notin [1..8]$
(2)	$f(5) \approx 8$
(3)	$f(8) \approx 5$

Unsatisfiable

Sub-domain [1..7] and critical point 8 can be found by binary search

Repair with 8 as next exception

Our Method - Second Round

Given clause set $N[\Delta_x]$

(1) $f(x) \not\approx x \vee x \notin [1..1000]$

(2) $f(5) \approx 8$

(3) $f(8) \approx 5$

Current set of exception points $\Pi_x \subseteq \Delta_x$

$\Pi_x = \{8\}$

Finite Domain Transformation $M = FD(N[\Delta_x], \Pi_x)$

(f)	$f(x) \approx \alpha_0 \vee x \notin [1..1000] \setminus \{8\}$	default interpretation for $f(x)$ in (1)
(f8)	$f(8) \approx \alpha_8$	f at exception point 8
(1f)	$\alpha_0 \not\approx x \vee x \notin [1..1000] \setminus \{8\}$	(f) applied to (1)
(1f8)	$\alpha_8 \not\approx 8$	(f8) applied to (1)
(2)	$f(5) \approx 8$	
(3)	$f(8) \approx 5$	

Satisfiable with $\alpha_0 \mapsto 8$, $\alpha_8 \mapsto 5$. Done

General Method: checkSAT/find

```
1   algorithm checkSAT(N[Δ_x])
2   // returns "satisfiable" or "unsatisfiable"
3   var Π_x := ∅_x // The current set of exceptions
4   while true {
5     let M = FD(N, Π_x)
6     if M is satisfiable return "satisfiable"
7     if M[∅_x] is unsatisfiable return "unsatisfiable"
8     let (x, d) = find(M)
9     Π_x := Π_x[x ↦ Π_x ∪ {d}]
10  }
```

Tacitly assume these checks are effective

Line 7 example, $\Pi_x = \{8\}$

(1) $f(x) > x \lor x \notin \Delta_x$

(2) $f(5) \approx 8$

(3) $f(8) \approx 5$

$\xrightarrow{\text{FD}}$

$\cancel{M}$ $\quad$ $M[\emptyset_x]$

$\cancel{f(x) \approx \alpha_0 \lor x \not\in \Delta_x \setminus \{8\}}$ $\quad$ $f(5) \approx 8$

$f(8) \approx \alpha_8$ $\quad$ $f(8) \approx 5$

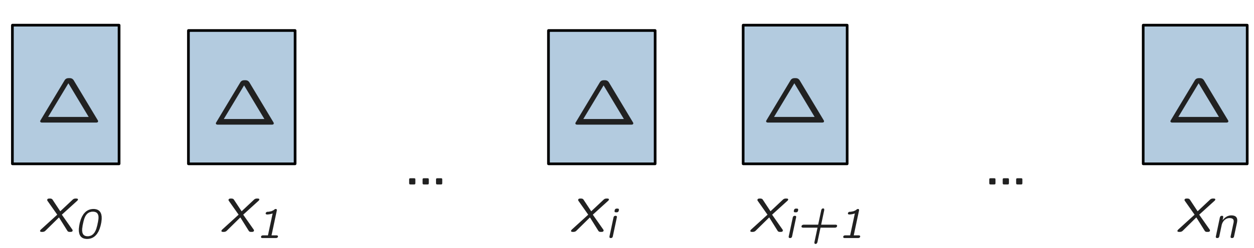
$\cancel{\alpha_0 > x \lor x \not\in \Delta_x \setminus \{8\}}$

$\alpha_8 > 8$

General Method: checkSat/find

```
algorithm find(M[Δ_x])
// returns a pair (x, d) such that x ∈ x and d ∈ Δ_x \ Π_x
let (x_1, ..., x_n) = x
for i = 1 to n {
  if M[∅_(x_1,...,x_i) · Δ_(x_{i+1},...,x_n)] is satisfiable {
     let Γ ⊆ Δ_{x_i} and d ∈ Γ such that
        M[∅_(x_1,...,x_{i-1}) · Γ_{x_i} · Δ_(x_{i+1},...,x_n)] is unsatisfiable and
        M[∅_(x_1,...,x_{i-1}) · (Γ \ {d})_{x_i} · Δ_(x_{i+1},...,x_n)] is satisfiable
     return (x_i, d)
  }
}
```

We know:

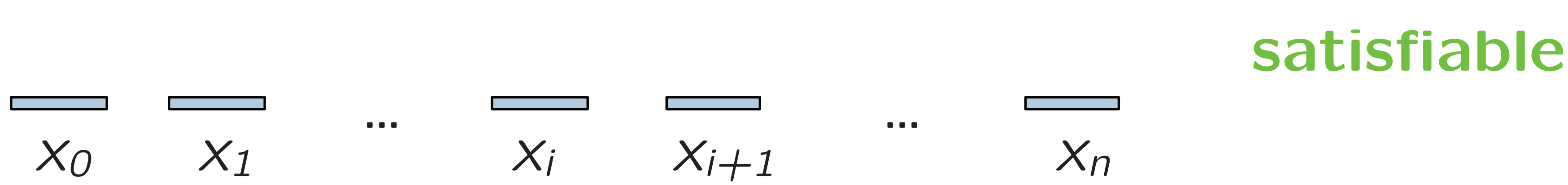


unsatisfiable

General Method: checkSat/find

algorithm find($M[\varDelta_{\mathbf{x}}]$)
// returns a pair (x, d) such that $x \in \mathbf{x}$ and $d \in \varDelta_x \setminus \varPi_x$
let $(x_1, \ldots, x_n) = \mathbf{x}$
for $i = 1$ **to** n $\{$
  if $M[\emptyset_{(x_1,\ldots,x_i)} \cdot \varDelta_{(x_{i+1},\ldots,x_n)}]$ is satisfiable $\{$
    let $\varGamma \subseteq \varDelta_{x_i}$ and $d \in \varGamma$ such that
      $M[\emptyset_{(x_1,\ldots,x_{i-1})} \cdot \varGamma_{x_i} \cdot \varDelta_{(x_{i+1},\ldots,x_n)}]$ is unsatisfiable and
      $M[\emptyset_{(x_1,\ldots,x_{i-1})} \cdot (\varGamma \setminus \{d\})_{x_i} \cdot \varDelta_{(x_{i+1},\ldots,x_n)}]$ is satisfiable
    return (x_i, d)
    $\}$
$\}$

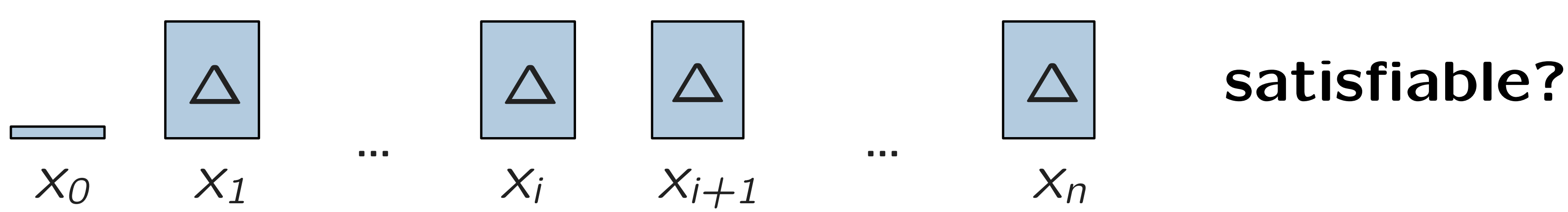
We know:



General Method: checkSat/find

algorithm find($M[\Delta_{\mathbf{x}}]$)
// returns a pair (x, d) such that $x \in \mathbf{x}$ and $d \in \Delta_x \setminus \Pi_x$
let $(x_1, \ldots, x_n) = \mathbf{x}$
for $i = 1$ **to** n {
  if $M[\emptyset_{(x_1,\ldots,x_i)} \cdot \Delta_{(x_{i+1},\ldots,x_n)}]$ is satisfiable {
    let $\Gamma \subseteq \Delta_{x_i}$ and $d \in \Gamma$ such that
      $M[\emptyset_{(x_1,\ldots,x_{i-1})} \cdot \Gamma_{x_i} \cdot \Delta_{(x_{i+1},\ldots,x_n)}]$ is unsatisfiable and
      $M[\emptyset_{(x_1,\ldots,x_{i-1})} \cdot (\Gamma \setminus \{d\})_{x_i} \cdot \Delta_{(x_{i+1},\ldots,x_n)}]$ is satisfiable
    return (x_i, d)
    }
}

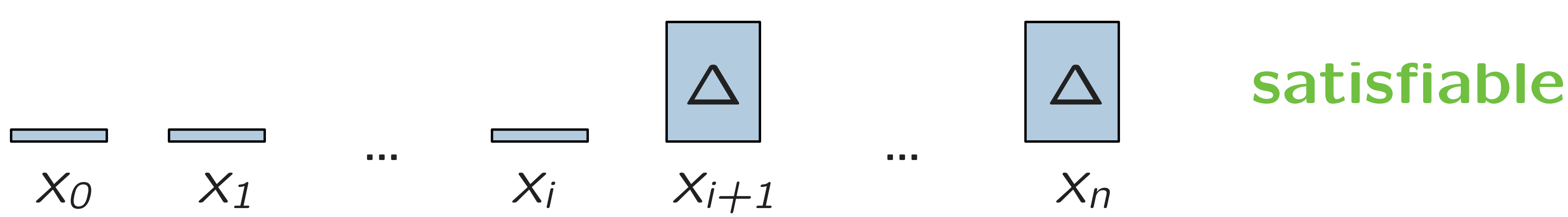
Search:



General Method: checkSat/find

algorithm find($M[\Delta_{\mathbf{x}}]$)
// returns a pair (x, d) such that $x \in \mathbf{x}$ and $d \in \Delta_x \setminus \Pi_x$
let $(x_1, \ldots, x_n) = \mathbf{x}$
for $i = 1$ **to** n {
  if $M[\emptyset_{(x_1,\ldots,x_i)} \cdot \Delta_{(x_{i+1},\ldots,x_n)}]$ is satisfiable {
    let $\Gamma \subseteq \Delta_{x_i}$ and $d \in \Gamma$ such that
      $M[\emptyset_{(x_1,\ldots,x_{i-1})} \cdot \Gamma_{x_i} \cdot \Delta_{(x_{i+1},\ldots,x_n)}]$ is unsatisfiable and
      $M[\emptyset_{(x_1,\ldots,x_{i-1})} \cdot (\Gamma \setminus \{d\})_{x_i} \cdot \Delta_{(x_{i+1},\ldots,x_n)}]$ is satisfiable
    return (x_i, d)
    }
}

Search:

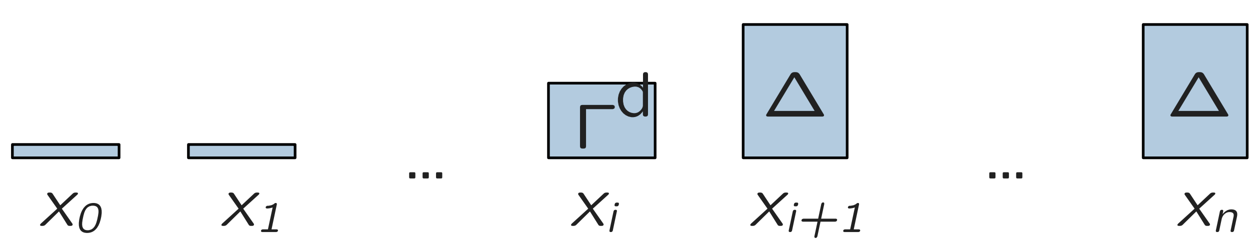


General Method: checkSat/find

algorithm find($M[\Delta_{\mathbf{x}}]$)
// returns a pair (x, d) such that $x \in \mathbf{x}$ and $d \in \Delta_x \setminus \Pi_x$
let $(x_1, \ldots, x_n) = \mathbf{x}$
for $i = 1$ **to** n $\{$
  if $M[\emptyset_{(x_1,\ldots,x_i)} \cdot \Delta_{(x_{i+1},\ldots,x_n)}]$ is satisfiable $\{$
    let $\Gamma \subseteq \Delta_{x_i}$ and $d \in \Gamma$ such that
      $M[\emptyset_{(x_1,\ldots,x_{i-1})} \cdot \Gamma_{x_i} \cdot \Delta_{(x_{i+1},\ldots,x_n)}]$ is unsatisfiable and
      $M[\emptyset_{(x_1,\ldots,x_{i-1})} \cdot (\Gamma \setminus \{d\})_{x_i} \cdot \Delta_{(x_{i+1},\ldots,x_n)}]$ is satisfiable
    return (x_i, d)
  $\}$
$\}$

Use binary search on Δ_x

Search:



unsatisfiable ("just")

Main Result

Assume that HSP decides satisfiability of clause sets $M = \mathrm{FD}(N[\mathbf{\Delta}_x], \mathbf{\Pi}_x)$

Theorem

For any set N of finitely quantified clauses, checkSAT(N) terminates with the correct result "satisfiable" or "unsatisfiable" for N.

Moreover, if the result is "unsatisfiable" then the non-domain restricted version of N is unsatisfiable, which is obtained from N by removing from all clauses in N all domain declarations $x \notin \mathbf{\Delta}_x$. $\square$

(1) $\mathsf{f}(x) > x \cancel{\vee x \notin \mathbf{\Delta}_x}$

(2) $\mathsf{f}(5) \approx 8$

(3) $\mathsf{f}(8) \approx 5$

Some Experiments

Array Example

$$\mathrm{read}(\mathrm{write}(a, i, x), i) \approx x$$

$$\mathrm{read}(\mathrm{write}(a, i, x), j) \approx \mathrm{read}(a, j) \vee i \approx j$$

$$\mathrm{read}(\mathrm{a}, i) \leq \mathrm{read}(\mathrm{a}, j) \vee \neg(i < j) \vee$$
$$\quad i \notin [1..1000] \vee j \notin [1..1000]$$

$$1 \leq \mathrm{m} \wedge \mathrm{m} < 1000$$

$$\mathrm{read}(\mathrm{a}, \mathrm{m}) < \mathrm{read}(\mathrm{a}, \mathrm{m}{+}1)$$

$\lvert\Delta\rvert$	#Iter	#TP	Time
10	3	15	2.3
20	3	17	2.6
50	3	19	2.8
100	3	21	2.8
200	3	23	2.8
500	3	25	2.9
1000	3	27	3.0
2000	3	29	3.0
5000	3	33	3.5

$m = 2$ variable occurrences
$n = 1000$ size of (largest) domain
Each iteration requires about $m + \mathrm{ld}(n) = 2 + 10$ prover calls in find

By contrast, ground instantiation gives $n^m = 10^6$ instances

Some Experiments

Running Example

$$\mathsf{f}(x) \not\approx x \vee x \notin \Delta$$

$$\mathsf{f}(5) \approx 8$$

$$\mathsf{f}(8) \approx 5$$

$\lvert\Delta\rvert$	**#Iter**	**#TP**	**Time**
10	2	5	<1
20	2	6	<1
50	2	8	<1
100	2	9	<1
200	2	10	<1
500	2	11	<1
1000	2	12	<1
2000	2	13	<1
5000	2	15	<1

Some Experiments

#	Problem	$\|\Delta\|$	#Iter	#TP	Time
1	$\mathsf{f}(x) > 1 + y \vee y < 0 \vee x \notin \Delta$	any	1	1	<1
2	$\mathsf{g}(x) \approx x \vee \mathsf{g}(x) \approx x + 1 \vee \neg(x \geq 0)$	10	9	32	5.5
	$\mathsf{g}(x) \approx -x \vee \neg(x < 0)$	20	20	86	55
	$\mathsf{f}(x) < \mathsf{g}(x) \vee x \notin \Delta$				
3	$\mathsf{f}(x_1, x_2, x_3, x_4) > x_1 + x_2 + x_3 + x_4 \vee$ $x_1 \notin \Delta \vee x_2 \notin \Delta \vee x_3 \notin \Delta \vee x_4 \notin \Delta$	any	1	1	<1

Problem 1: default interpretation enough

Problem 2: have sufficient completeness wrt. g, need to treat only f

Problem 3: default interpretation enough

Z3: does not solve problems 1 and 2, solves problem 3 up to $|\Delta| = 60$

Z3: solves running examples above

Conclusions

Presented a method on top of HSP for theorem (dis)proving under finitely quantified variables

Main idea: conflict-driven repair of default interpretation

Requires BG reasoner for EA-fragment

Meant to scale well with domain size

However worst case needs exceptions "everywhere"

Can (sometimes) be used on top of SMT

Eliminate first non-ground definitions by exhaustive superposition

Generalizes instantiation heuristics known from SMT

Return "unsatisfiable" or "satisfiable (over finite domain)"

(Supposing underlying prover terminates)

Future work

Instantiation-based methods as special case of the method here?