CDCL as Saturation

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Research Interest: Automated Deduction

Logics

First-order logic with equality / EPR

Theory reasoning, e.g., modulo LIA

CTL*(FO(LIA))

Calculi

Model evolution (first-order DPLL)

Hierarchic superposition

Reasoning services

Proving theorems

Disproving theorems

Systems

Darwin

Beagle   Fitzroy

Example: Theory Reasoning

Theory reasoning

Lists over integers

$(l \approx \mathrm{nil}) \vee (l \approx \mathrm{cons}(\mathrm{head}(l), \mathrm{tail}(l)))$

$\neg(\mathrm{cons}(k, l) \approx \mathrm{nil})$

$\mathrm{head}(\mathrm{cons}(k, l)) \approx k$

$\mathrm{tail}(\mathrm{cons}(k, l)) \approx l$

The **inRange** predicate, e.g. $\mathrm{inRange}([1{,}0{,}5], 6)$

$\mathrm{nRange}(l, n) \leftrightarrow (l \approx \mathrm{nil} \vee (0 \leq \mathrm{head}(l) < n \wedge \mathrm{inRange}(\mathrm{tail}(l), n)))$

Conjecture

$\forall\ \mathrm{l{:}list}\ \mathrm{n{:}int}\ (\neg(\mathrm{l} \approx \mathrm{nil}) \rightarrow (\mathrm{inRange}(\mathrm{l}, \mathrm{n}) \rightarrow \mathrm{inRange}(\mathrm{cons}(\mathrm{head}(\mathrm{l}), \mathrm{l}), \mathrm{n})))$

LIA + Lists/Arrays + Hypotheses $\models$ Conjecture ?

LIA + Lists/Arrays + Hypotheses $\not\models$ Conjecture ?

Example: Theory Reasoning

Theory reasoning

Lists over integers

$(l \approx \mathrm{nil}) \vee (l \approx \mathrm{cons}(\mathrm{head}(l), \mathrm{tail}(l)))$

$\neg(\mathrm{cons}(k, l) \approx \mathrm{nil})$

$\mathrm{head}(\mathrm{cons}(k, l)) \approx k$

$\mathrm{tail}(\mathrm{cons}(k, l)) \approx l$

Injective

Not surjective

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Conjecture

$\forall\ \mathrm{l{:}list}\ \mathrm{n{:}int}\ (\neg(\mathrm{l} \approx \mathrm{nil}) \rightarrow (\mathrm{inRange}(\mathrm{l}, \mathrm{n}) \rightarrow \mathrm{inRange}(\mathrm{cons}(\mathrm{head}(\mathrm{l}), \mathrm{l}), \mathrm{n})))$

LIA + Lists/Arrays + Hypotheses $\models$ Conjecture ?

LIA + Lists/Arrays + Hypotheses $\not\models$ Conjecture ?

Example: Theory Reasoning

Theory reasoning

Lists over integers

$(l \approx \mathrm{nil}) \vee (l \approx \mathrm{cons}(\mathrm{head}(l), \mathrm{tail}(l)))$

$\neg(\mathrm{cons}(k, l) \approx \mathrm{nil})$

$\mathrm{head}(\mathrm{cons}(k, l)) \approx k$

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Injective

Not surjective

The **inRange** predicate, e.g. **inRange([1,0,5], 6)**

$\mathrm{nRange}(l, n) \leftrightarrow (l \approx \mathrm{nil} \vee (0 \leq \mathrm{head}(l) < n \wedge \mathrm{inRange}(\mathrm{tail}(l), n)))$

Conjecture

$\forall\ \mathrm{l{:}list\ n{:}int}\ (\neg(\mathrm{l} \approx \mathrm{nil}) \rightarrow (\mathrm{inRange}(\mathrm{l}, \mathrm{n}) \rightarrow \mathrm{inRange}(\mathrm{cons}(\mathrm{head}(\mathrm{l}), \mathrm{l}), \mathrm{n})))$

LIA + Lists/A **es $\models$ Conjecture ?**

“Proving infinite satisfiability”

LIA + Lists/Arrays + Hypotheses $\not\models$ Conjecture ?

CDCL as Saturation - Motivation

Background

Conflict driven clause learning (**CDCL**) for building SAT solvers

Superposition/Resolution (**Saturation**) for building FOL theorem provers

This talk

Modelling the essence of CDCL in a saturation based framework

Technical difficulty: modelling context switches (backjumping)

Goals

Scientific curiosity: relationship between CDCL and saturation?

(Building SAT solvers)

Building FO theorem provers

- Instance-based methods: [Plaisted], [Korovin], [BTinelli],...
- More recently: [AlagiWeidenbach], [BonacinaPlaisted]

Example CDCL Derivation

$$F \lor D \tag{0}$$

$$G \lor E \lor \neg D \tag{1}$$

$$B \lor \neg A \tag{2}$$

$$C \lor \neg B \tag{3}$$

$$\neg D \lor \neg C \lor \neg A \tag{4}$$

$$\neg C \lor A \tag{5}$$

$$C \lor A \tag{6}$$

$$D \lor \neg F \tag{7}$$

Example CDCL Derivation

$F \vee D$ (0)

$G \vee E \vee \neg D$ (1)

$B \vee \neg A$ (2)

$C \vee \neg B$ (3)

$\neg D \vee \neg C \vee \neg A$ (4)

$\neg C \vee A$ (5)

$C \vee A$ (6)

$D \vee \neg F$ (7)

$[D]$ (Decide)

Example CDCL Derivation

$F \lor D$ (0)

$G \lor E \lor \neg D$ (1)

$B \lor \neg A$ (2)

$C \lor \neg B$ (3)

$\neg D \lor \neg C \lor \neg A$ (4)

$\neg C \lor A$ (5)

$C \lor A$ (6)

$D \lor \neg F$ (7)

[D] (Decide)

[E] (Decide)

Example CDCL Derivation

$F \lor D$ (0)

$G \lor E \lor \neg D$ (1)

$B \lor \neg A$ (2)

$C \lor \neg B$ (3)

$\neg D \lor \neg C \lor \neg A$ (4)

$\neg C \lor A$ (5)

$C \lor A$ (6)

$D \lor \neg F$ (7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)

Example CDCL Derivation

$F \lor D$ (0)

$G \lor E \lor \neg D$ (1)

$B \lor \neg A$ (2)

$C \lor \neg B$ (3)

$\neg D \lor \neg C \lor \neg A$ (4)

$\neg C \lor A$ (5)

$C \lor A$ (6)

$D \lor \neg F$ (7)

[D] (Decide)

[E] (Decide)

[A] (Decide)

B (by 2,[A])

Example CDCL Derivation

$F \lor D$	(0)	$\neg D \lor \neg C \lor \neg A$	(4)
$G \lor E \lor \neg D$	(1)	$\neg C \lor A$	(5)
$B \lor \neg A$	(2)	$C \lor A$	(6)
$C \lor \neg B$	(3)	$D \lor \neg F$	(7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)

Example CDCL Derivation

$F \lor D$	(0)	$\neg D \lor \neg C \lor \neg A$	(4)
$G \lor E \lor \neg D$	(1)	$\neg C \lor A$	(5)
$B \lor \neg A$	(2)	$C \lor A$	(6)
$C \lor \neg B$	(3)	$D \lor \neg F$	(7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)
	(conflict 4)

Example CDCL Derivation

$F \lor D$	(0)
$G \lor E \lor \neg D$	(1)
$B \lor \neg A$	(2)
$C \lor \neg B$	(3)
$\neg D \lor \neg C \lor \neg A$	(4)
$\neg C \lor A$	(5)
$C \lor A$	(6)
$D \lor \neg F$	(7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)
	(conflict 4)

$\neg D \lor \neg B \lor \neg A$ (8 by 4,3)

Example CDCL Derivation

$F \vee D$	(0)	$\neg D \vee \neg C \vee \neg A$	(4)
$G \vee E \vee \neg D$	(1)	$\neg C \vee A$	(5)
$B \vee \neg A$	(2)	$C \vee A$	(6)
$C \vee \neg B$	(3)	$D \vee \neg F$	(7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)
	(conflict 4)

$\neg D \vee \neg B \vee \neg A$	(8 by 4,3)
$\neg D \vee \neg A$	(9 by 8,2)

Example CDCL Derivation

$F \lor D$	(0)	$\neg D \lor \neg C \lor \neg A$	(4)
$G \lor E \lor \neg D$	(1)	$\neg C \lor A$	(5)
$B \lor \neg A$	(2)	$C \lor A$	(6)
$C \lor \neg B$	(3)	$D \lor \neg F$	(7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)
	(conflict 4)

Backjump

$\neg D \lor \neg B \lor \neg A$	(8 by 4,3)
$\neg D \lor \neg A$	(9 by 8,2)

Example CDCL Derivation

$F \vee D$	(0)	$\neg D \vee \neg C \vee \neg A$	(4)
$G \vee E \vee \neg D$	(1)	$\neg C \vee A$	(5)
$B \vee \neg A$	(2)	$C \vee A$	(6)
$C \vee \neg B$	(3)	$D \vee \neg F$	(7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)
	(conflict 4)

Backjump

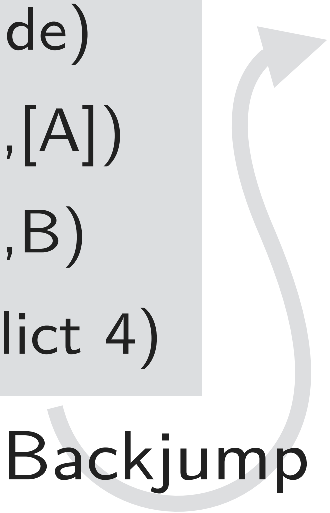
[D]

$\neg D \vee \neg B \vee \neg A$	(8 by 4,3)
$\neg D \vee \neg A$	(9 by 8,2)

Example CDCL Derivation

$F \lor D$	(0)	$\neg D \lor \neg C \lor \neg A$	(4)
$G \lor E \lor \neg D$	(1)	$\neg C \lor A$	(5)
$B \lor \neg A$	(2)	$C \lor A$	(6)
$C \lor \neg B$	(3)	$D \lor \neg F$	(7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)
	(conflict 4)



[D]
[E]

$\neg D \lor \neg B \lor \neg A$	(8 by 4,3)
$\neg D \lor \neg A$	(9 by 8,2)

Example CDCL Derivation

$F \lor D$	(0)	$\neg D \lor \neg C \lor \neg A$	(4)
$G \lor E \lor \neg D$	(1)	$\neg C \lor A$	(5)
$B \lor \neg A$	(2)	$C \lor A$	(6)
$C \lor \neg B$	(3)	$D \lor \neg F$	(7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)
	(conflict 4)

Backjump

[D]	
[E]	
$\neg A$	(by 9,[D])

$\neg D \lor \neg B \lor \neg A$	(8 by 4,3)
$\neg D \lor \neg A$	(9 by 8,2)

Example CDCL Derivation

$F \lor D$	(0)	$\neg D \lor \neg C \lor \neg A$	(4)
$G \lor E \lor \neg D$	(1)	$\neg C \lor A$	(5)
$B \lor \neg A$	(2)	$C \lor A$	(6)
$C \lor \neg B$	(3)	$D \lor \neg F$	(7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)
	(conflict 4)

Backjump

[D]	
[E]	
$\neg A$	(by 9,[D])
C	(by 6,$\neg$A)

$\neg D \lor \neg B \lor \neg A$	(8 by 4,3)
$\neg D \lor \neg A$	(9 by 8,2)

Example CDCL Derivation

$F \lor D$	(0)	$\neg D \lor \neg C \lor \neg A$	(4)
$G \lor E \lor \neg D$	(1)	$\neg C \lor A$	(5)
$B \lor \neg A$	(2)	$C \lor A$	(6)
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[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)
	(conflict 4)

[D]	
[E]	
$\neg A$	(by 9,[D])
C	(by 6,$\neg$A)
	(conflict 5)

Backjump

$\neg D \lor \neg B \lor \neg A$	(8 by 4,3)
$\neg D \lor \neg A$	(9 by 8,2)

Example CDCL Derivation

$F \vee D$	(0)	$\neg D \vee \neg C \vee \neg A$	(4)
$G \vee E \vee \neg D$	(1)	$\neg C \vee A$	(5)
$B \vee \neg A$	(2)	$C \vee A$	(6)
$C \vee \neg B$	(3)	$D \vee \neg F$	(7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)
	(conflict 4)

Backjump

[D]	
[E]	
$\neg A$	(by 9,[D])
C	(by 6,$\neg$A)
	(conflict 5)

$\neg D \vee \neg B \vee \neg A$	(8 by 4,3)
$\neg D \vee \neg A$	(9 by 8,2)

A	(10 by 6,5)

Example CDCL Derivation

$F \lor D$	(0)	$\neg D \lor \neg C \lor \neg A$	(4)
$G \lor E \lor \neg D$	(1)	$\neg C \lor A$	(5)
$B \lor \neg A$	(2)	$C \lor A$	(6)
$C \lor \neg B$	(3)	$D \lor \neg F$	(7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)
	(conflict 4)

Backjump

[D]	
[E]	
$\neg A$	(by 9,[D])
C	(by 6,$\neg$A)
	(conflict 5)

$\neg D \lor \neg B \lor \neg A$	(8 by 4,3)
$\neg D \lor \neg A$	(9 by 8,2)

A	(10 by 6,5)
$\neg D$	(11 by 10,9)

Example CDCL Derivation

$F \lor D$	(0)	$\neg D \lor \neg C \lor \neg A$	(4)
$G \lor E \lor \neg D$	(1)	$\neg C \lor A$	(5)
$B \lor \neg A$	(2)	$C \lor A$	(6)
$C \lor \neg B$	(3)	$D \lor \neg F$	(7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)
	(conflict 4)

Backjump

$\neg D \lor \neg B \lor \neg A$	(8 by 4,3)
$\neg D \lor \neg A$	(9 by 8,2)

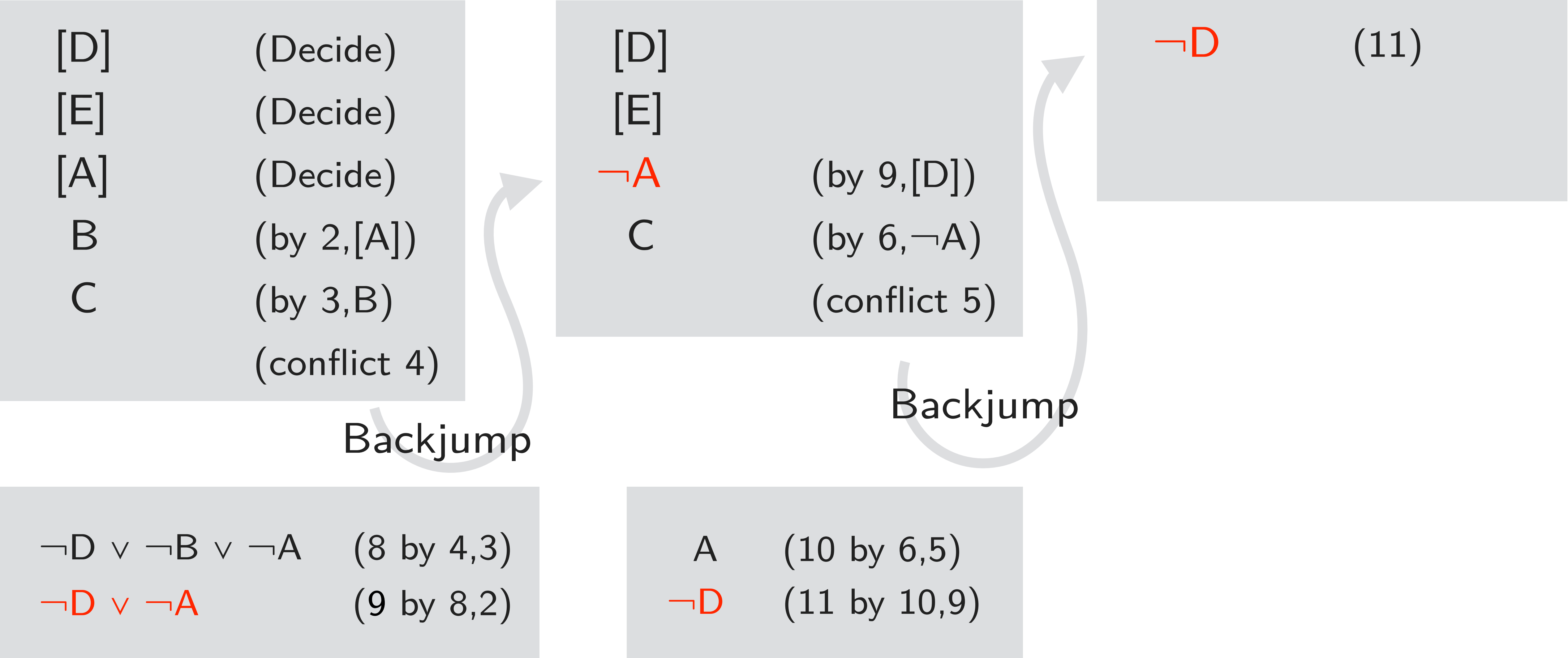
[D]	
[E]	
$\neg A$	(by 9,[D])
C	(by 6,$\neg$A)
	(conflict 5)

Backjump

A	(10 by 6,5)
$\neg D$	(11 by 10,9)

Example CDCL Derivation

F ∨ D	(0)	¬D ∨ ¬C ∨ ¬A	(4)
G ∨ E ∨ ¬D	(1)	¬C ∨ A	(5)
B ∨ ¬A	(2)	C ∨ A	(6)
C ∨ ¬B	(3)	D ∨ ¬F	(7)



[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)
	(conflict 4)

¬D ∨ ¬B ∨ ¬A	(8 by 4,3)
¬D ∨ ¬A	(9 by 8,2)

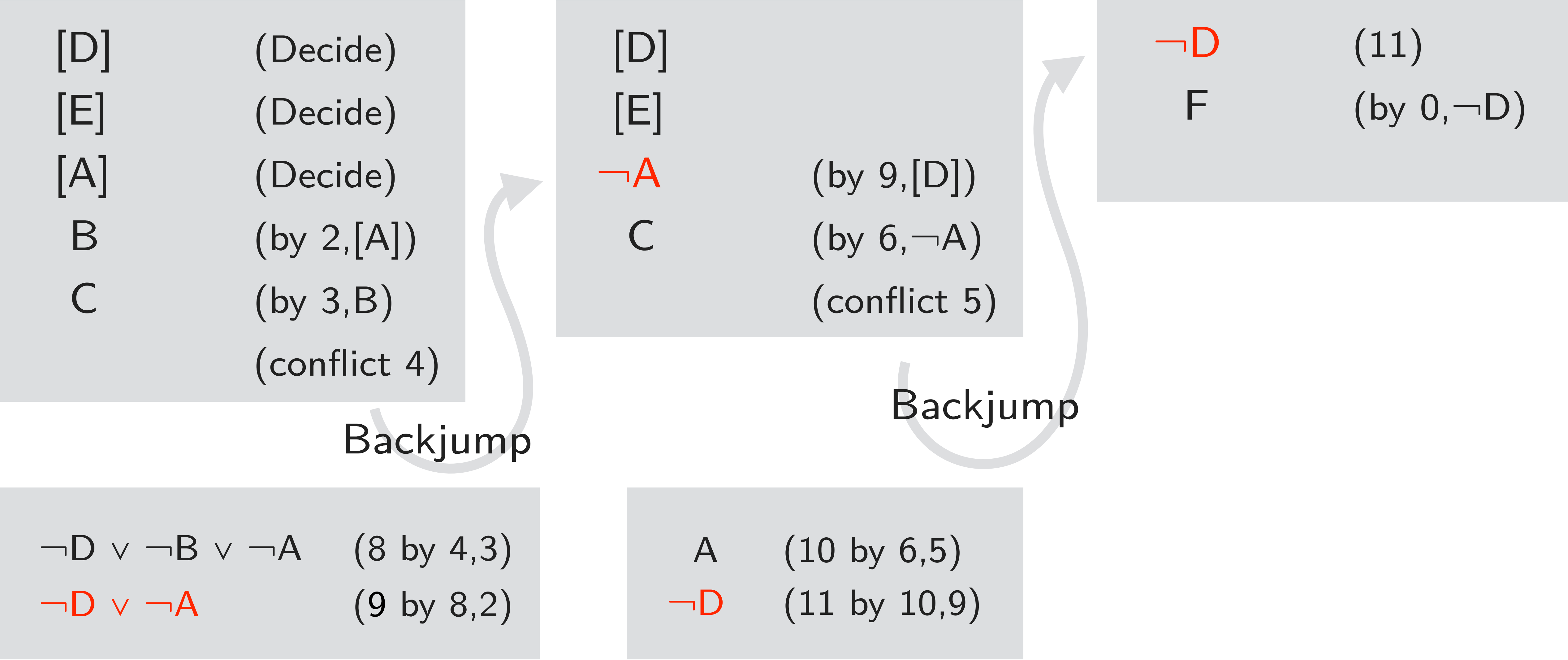
[D]	
[E]	
¬A	(by 9,[D])
C	(by 6,¬A)
	(conflict 5)

A	(10 by 6,5)
¬D	(11 by 10,9)

¬D	(11)

Example CDCL Derivation

$F \vee D$	(0)	$\neg D \vee \neg C \vee \neg A$	(4)
$G \vee E \vee \neg D$	(1)	$\neg C \vee A$	(5)
$B \vee \neg A$	(2)	$C \vee A$	(6)
$C \vee \neg B$	(3)	$D \vee \neg F$	(7)



[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)
	(conflict 4)

Backjump

$\neg D \vee \neg B \vee \neg A$	(8 by 4,3)
$\neg D \vee \neg A$	(9 by 8,2)

[D]	
[E]	
$\neg A$	(by 9,[D])
C	(by 6,$\neg$A)
	(conflict 5)

Backjump

A	(10 by 6,5)
$\neg D$	(11 by 10,9)

$\neg D$	(11)
F	(by 0,$\neg$D)

Example CDCL Derivation

F ∨ D	(0)	¬D ∨ ¬C ∨ ¬A	(4)
G ∨ E ∨ ¬D	(1)	¬C ∨ A	(5)
B ∨ ¬A	(2)	C ∨ A	(6)
C ∨ ¬B	(3)	D ∨ ¬F	(7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
B	(by 2,[A])
C	(by 3,B)
	(conflict 4)

Backjump

¬D ∨ ¬B ∨ ¬A	(8 by 4,3)
¬D ∨ ¬A	(9 by 8,2)

[D]	
[E]	
¬A	(by 9,[D])
C	(by 6,¬A)
	(conflict 5)

Backjump

A	(10 by 6,5)
¬D	(11 by 10,9)

¬D	(11)
F	(by 0,¬D)
$\square$	(FAIL 7)

CDCL as Saturation - Alternatives

(1) Nothing to be done

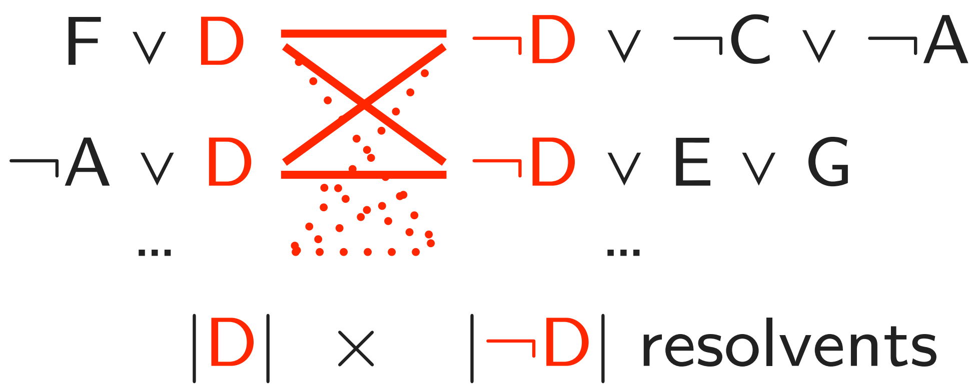
Every CDCL refutation induces a resolution refutation

⇒ Closure under resolution inferences will find that refutation

Problem: ignores search space

Clause recombination problem [Plaisted]:

Resolution



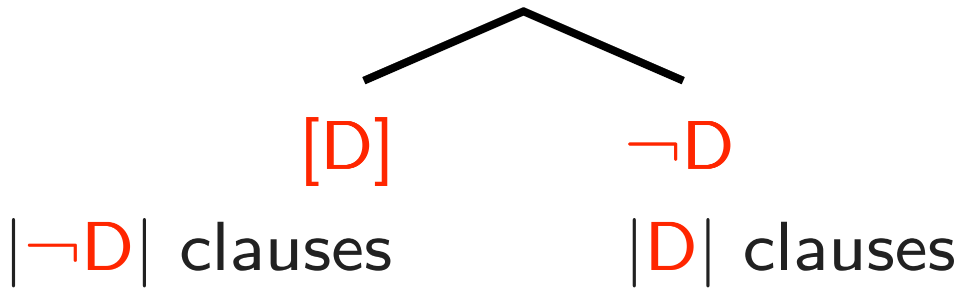
$F \lor D$ $\neg D \lor \neg C \lor \neg A$

$\neg A \lor D$ $\neg D \lor E \lor G$

... ...

$|D| \times |\neg D|$ resolvents

CDCL

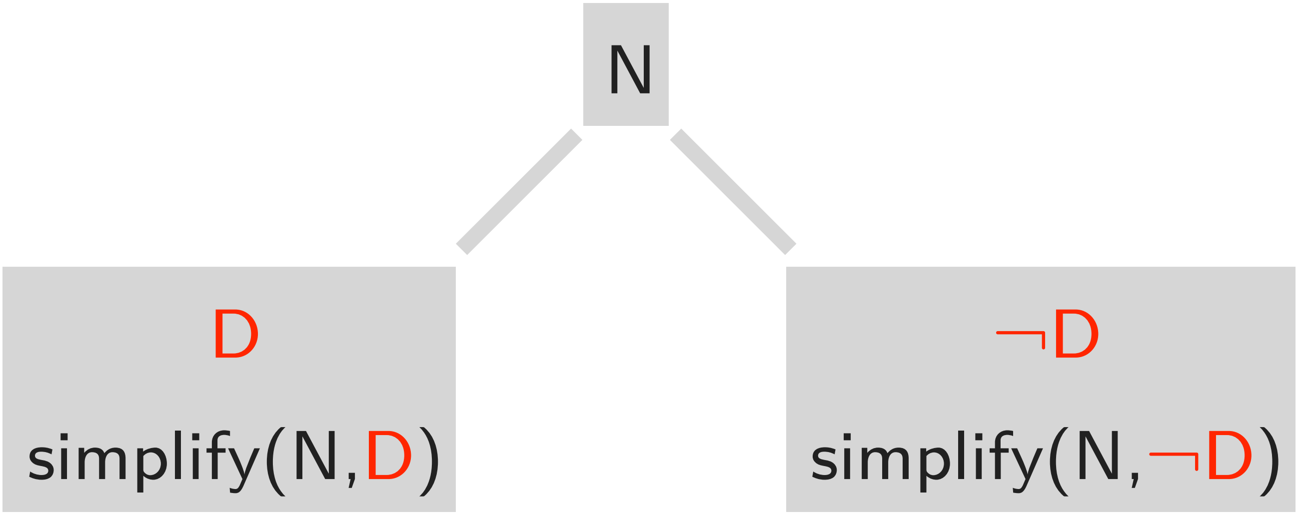


$[D]$ $\neg D$

$|\neg D|$ clauses $|D|$ clauses

CDCL as Saturation - Alternatives

(2) Add a split rule to resolution



N

D

simplify(N,D)

$\neg$D

simplify(N,$\neg$D)

Problems

Nothing new

Lifting to first-order logic? (But see Model Evolution)

(3) Approach taken here

Explained in the rest of this talk

Data Structures

Data Structures

Syntax

Decision literal, e.g.,	$[A], [B]$
Constraint clause, e.g.,	$\neg D \vee C \leftarrow [B], [A]$
Ordinary clause, e.g.,	$\neg D \vee C \leftarrow$
Unit clause, e.g.,	$C \leftarrow [B], [A]$

State: $M \blacksquare N$ where

M is a set of *decision* literals

N is a set of constraint clauses, unit or ordinary

Data Structures

“Derivation of $\neg D \vee C$ depends on $[B]$ and $[A]$”

Syntax

Decision literal, e.g., $[A], [B]$

Constraint clause, e.g., $\neg D \vee C \leftarrow [B], [A]$

Ordinary clause, e.g., $\neg D \vee C \leftarrow$

Unit clause, e.g., $C \leftarrow [B], [A]$

State: $M \,\blacksquare\, N$ where

M is a set of *decision* literals

N is a set of constraint clauses, unit or ordinary

Data Structures

Syntax

Decision literal, e.g., $[A], [B]$

Constraint clause, e.g., $\neg D \lor C \leftarrow [B], [A]$

“Derivation of $\neg D \lor C$ depends on $[B]$ and $[A]$”

Ordinary clause, e.g., $\neg D \lor C \leftarrow$

Input clauses

Unit clause, e.g., $C \leftarrow [B], [A]$

State: $M \mathbin{\blacksquare} N$ where

M is a set of *decision* literals

N is a set of constraint clauses, unit or ordinary

Data Structures

Syntax

Decision literal, e.g., $[A], [B]$

Constraint clause, e.g., $\neg D \lor C \leftarrow [B], [A]$

“Derivation of $\neg D \lor C$ depends on $[B]$ and $[A]$”

Ordinary clause, e.g., $\neg D \lor C \leftarrow$

Input clauses

Unit clause, e.g., $C \leftarrow [B], [A]$

By Propagate/ Backjump

State: $M \mathbin{\blacksquare} N$ where

M is a set of *decision* literals

N is a set of constraint clauses, unit or ordinary

Data Structures

Syntax

Decision literal, e.g., $[A], [B]$

Constraint clause, e.g., $\neg D \vee C \leftarrow [B], [A]$

“Derivation of $\neg D \vee C$ depends on $[B]$ and $[A]$”

Ordinary clause, e.g., $\neg D \vee C \leftarrow$

Input clauses

Unit clause, e.g., $C \leftarrow [B], [A]$

By Propagate/ Backjump

State: $M \blacksquare N$ where

M is a set of *decision* literals

$\neq$ CDCL

N is a set of constraint clauses, unit or ordinary

Data Structures

Syntax

Decision literal, e.g., [A], [B]

Constraint clause, e.g., $\neg D \vee C \leftarrow [B], [A]$

"Derivation of $\neg D \vee C$ depends on [B] and [A]"

Ordinary clause, e.g., $\neg D \vee C \leftarrow$

Input clauses

Unit clause, e.g., $C \leftarrow [B], [A]$

By Propagate/ Backjump

State: M ▪ N where

M is a set of *decision* literals

≠ CDCL

N is a set of constraint clauses, unit or ordinary

Semantics

[C]	C is true by **default**
$\neg C \leftarrow [B], [A]$	C is false if A and B are true by default, **overriding** [C]
$C \leftarrow [B], [A]$	C is true if A and B are true by default, **overriding** [C]

Inference Rules by Example

Propagate

Decide

Backjump

Fail

Inference Rules by Example

Propagate

$$A \vee \neg B \vee \quad C \;\leftarrow$$

Decide

Backjump

Fail

Inference Rules by Example

Propagate

$$\neg C \leftarrow [D]$$

$$A \vee \neg B \vee C \leftarrow$$

Decide

Backjump

Fail

Inference Rules by Example

Propagate

$$\begin{array}{ccccccc} & & [B] & & \neg C & \leftarrow & [D] \\ A & \vee & \neg B & \vee & C & \leftarrow & \end{array}$$

Decide

Backjump

Fail

Inference Rules by Example

Propagate

$$\frac{\begin{array}{ccccccc} & & [B] & & \neg C & \leftarrow & [D] \\ A & \vee & \neg B & \vee & C & \leftarrow & \end{array}}{A \leftarrow [B],\ [D]}$$

Decide

Backjump

Fail

Inference Rules by Example

Propagate

Positive

$$\frac{\begin{array}{c} [B] \qquad \neg C \leftarrow [D] \\ A \lor \neg B \lor C \leftarrow \end{array}}{A \leftarrow [B], [D]}$$

Decide

Backjump

Fail

Inference Rules by Example

Propagate

Positive

$$\frac{\begin{array}{ccccc} & [B] & & \neg C & \leftarrow [D] \\ A \lor & \neg B & \lor & C & \leftarrow \end{array}}{A \leftarrow [B], [D]} \quad \text{if } A \succ \neg B \lor C$$

Decide

Backjump

Fail

Inference Rules by Example

Propagate

Positive

$$\frac{\begin{array}{ccccccc} & & [B] & & \neg C & \leftarrow & [D] \\ A & \lor & \neg B & \lor & C & \leftarrow & \end{array}}{A \leftarrow [B], [D]} \quad \text{if } A \succ \neg B \lor C$$

Decide

$$\neg E \lor A \lor \neg B \lor C \leftarrow$$

Backjump

Fail

Inference Rules by Example

Propagate

Positive

$$\frac{\begin{array}{ccccccc} & & [B] & & \neg C & \leftarrow & [D] \\ A & \lor & \neg B & \lor & C & \leftarrow & \end{array}}{A \leftarrow [B], [D]} \quad \text{if } A \succ \neg B \lor C$$

Decide

$$\begin{array}{ccccccccc} & & & & & & \neg C & \leftarrow & [D] \\ \neg E & \lor & A & \lor & \neg B & \lor & C & \leftarrow & \end{array}$$

Backjump

Fail

Inference Rules by Example

Propagate

Positive

$$\frac{\begin{array}{ccccc} & [B] & & \neg C & \leftarrow [D] \\ A \vee & \neg B & \vee & C & \leftarrow \end{array}}{A \leftarrow [B], [D]} \quad \text{if } A > \neg B \vee C$$

Decide

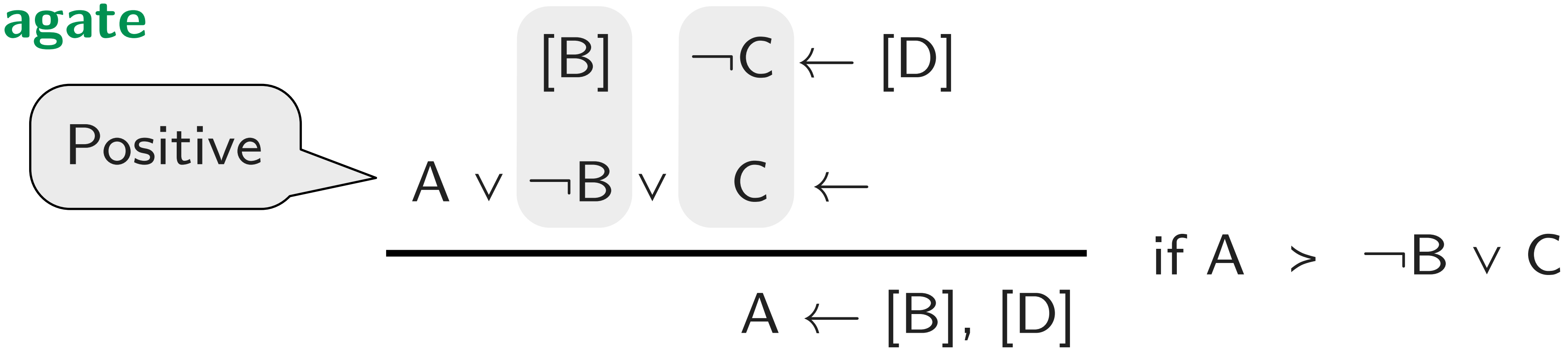
$$\begin{array}{ccccccc} & & & [B] & & \neg C & \leftarrow [D] \\ \neg E & \vee & A \vee & \neg B & \vee & C & \leftarrow \end{array}$$

Backjump

Fail

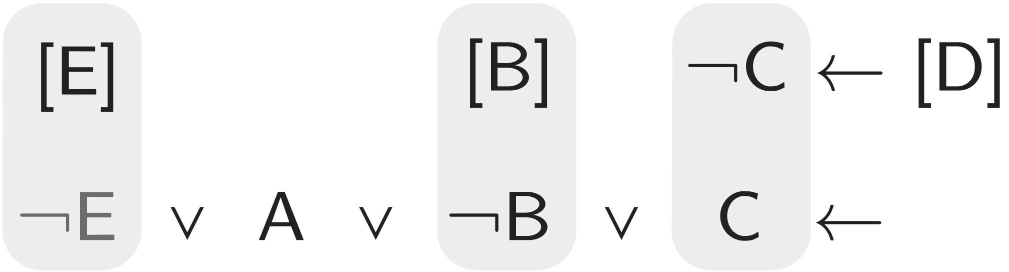
Inference Rules by Example

Propagate



$$\dfrac{\begin{array}{ccccccl} & & [B] & & \neg C & \leftarrow & [D] \\ A & \vee & \neg B & \vee & C & \leftarrow & \end{array}}{A \leftarrow [B],\ [D]} \quad \text{if } A > \neg B \vee C$$

Decide



$$\begin{array}{ccccccccl} [E] & & & & [B] & & \neg C & \leftarrow & [D] \\ \neg E & \vee & A & \vee & \neg B & \vee & C & \leftarrow & \end{array}$$

Backjump

Fail

Inference Rules by Example

Propagate

Positive

$$\frac{\begin{array}{cccccl} & & [B] & & \neg C & \leftarrow [D] \\ A & \vee & \neg B & \vee & C & \leftarrow \end{array}}{A \leftarrow [B], [D]} \quad \text{if } A \succ \neg B \vee C$$

Decide

$$\frac{\begin{array}{ccccccccl} [E] & & & & [B] & & \neg C & \leftarrow [D] \\ \neg E & \vee & A & \vee & \neg B & \vee & C & \leftarrow \end{array}}{[A]}$$

Backjump

Fail

Inference Rules by Example

Propagate

Positive

$$\frac{\begin{array}{ccccccc} & & [B] & & \neg C & \leftarrow & [D] \\ A & \vee & \neg B & \vee & C & \leftarrow & \end{array}}{A \leftarrow [B],\ [D]} \qquad \text{if } A \ > \ \neg B \vee C$$

Decide

$$\frac{\begin{array}{ccccccccc} [E] & & & & [B] & & \neg C & \leftarrow & [D] \\ \neg E & \vee & A & \vee & \neg B & \vee & C & \leftarrow & \end{array}}{[A]} \qquad \text{if } \neg E \ > \ A \ > \ \neg B \vee C$$

Backjump

Fail

Inference Rules by Example

Propagate

Positive

$$\frac{\begin{array}{cccccc} & & [B] & & \neg C & \leftarrow [D] \\ A & \vee & \neg B & \vee & C & \leftarrow \end{array}}{A \leftarrow [B], [D]} \quad \text{if } A > \neg B \vee C$$

Decide

$$\frac{\begin{array}{cccccccc} [E] & & & & [B] & & \neg C & \leftarrow [D] \\ \neg E & \vee & A & \vee & \neg B & \vee & C & \leftarrow \end{array}}{[A]} \quad \text{if } \neg E > A > \neg B \vee C$$

Backjump

Fail

$$\neg E \vee A \vee \neg B \vee C \leftarrow$$

Inference Rules by Example

Propagate

Positive

$$\frac{\begin{array}{ccccccc} & & [B] & & \neg C & \leftarrow & [D] \\ A & \lor & \neg B & \lor & C & \leftarrow & \end{array}}{A \leftarrow [B], [D]} \quad \text{if } A \;>\; \neg B \lor C$$

Decide

$$\frac{\begin{array}{ccccccccc} [E] & & & & [B] & & \neg C & \leftarrow & [D] \\ \neg E & \lor & A & \lor & \neg B & \lor & C & \leftarrow & \end{array}}{[A]} \quad \text{if } \neg E \;>\; A \;>\; \neg B \lor C$$

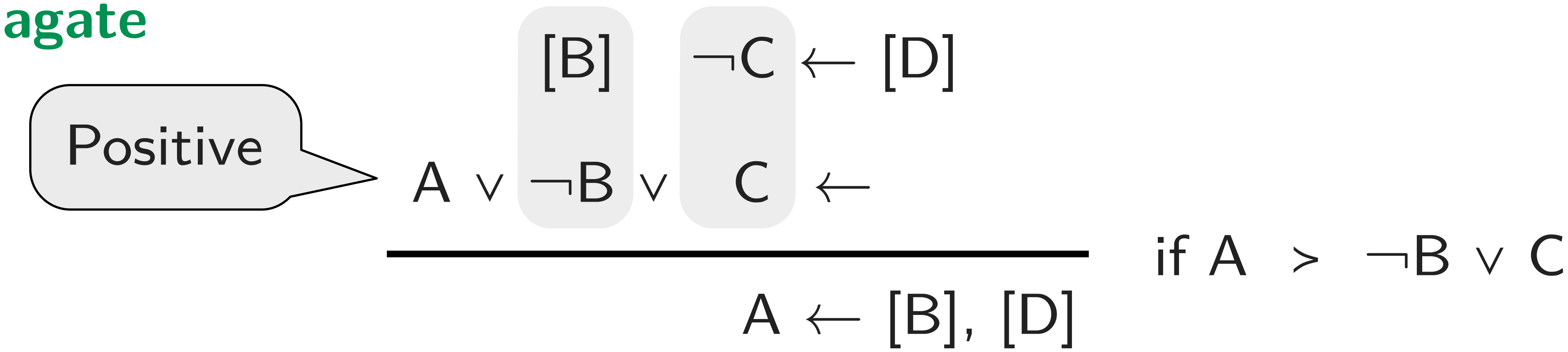
Backjump

Fail

$$\begin{array}{ccccccccc} & & & & & & \neg C & \leftarrow & [D] \\ \neg E & \lor & A & \lor & \neg B & \lor & C & \leftarrow & \end{array}$$

Inference Rules by Example

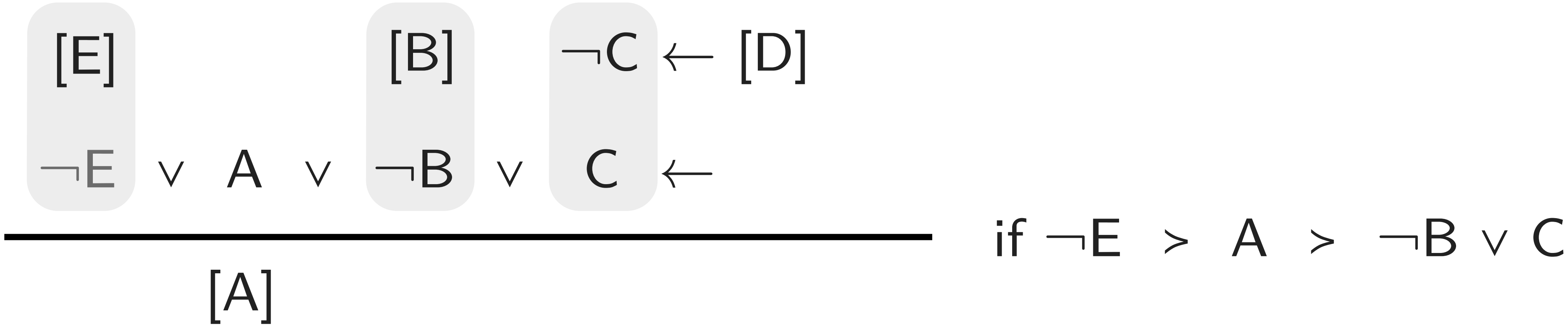
Propagate



Positive

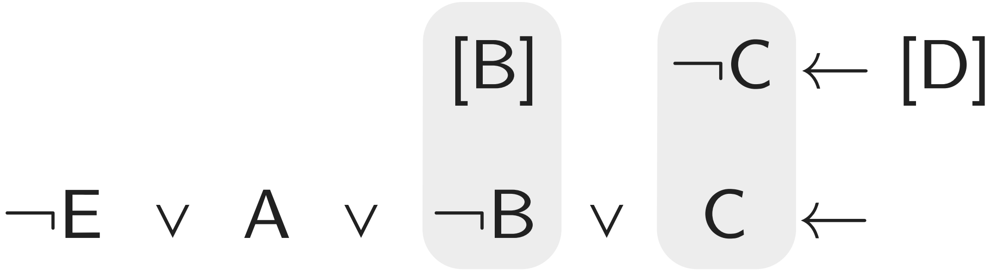
$$\frac{\begin{array}{ccccccl} & & [B] & & \neg C & \leftarrow & [D] \\ A & \lor & \neg B & \lor & C & \leftarrow & \end{array}}{A \leftarrow [B], [D]} \quad \text{if } A > \neg B \lor C$$

Decide



$$\frac{\begin{array}{cccccccccl} [E] & & & & [B] & & \neg C & \leftarrow & [D] \\ \neg E & \lor & A & \lor & \neg B & \lor & C & \leftarrow & \end{array}}{[A]} \quad \text{if } \neg E > A > \neg B \lor C$$

Backjump



$$\begin{array}{cccccccccl} & & & & [B] & & \neg C & \leftarrow & [D] \\ \neg E & \lor & A & \lor & \neg B & \lor & C & \leftarrow & \end{array}$$

Fail

Inference Rules by Example

Propagate

Positive

$$\dfrac{\begin{array}{ccccccl} & & [B] & & \neg C & \leftarrow & [D] \\ A & \vee & \neg B & \vee & C & \leftarrow & \end{array}}{A \leftarrow [B],\ [D]} \quad \text{if } A \ > \ \neg B \vee C$$

Decide

$$\dfrac{\begin{array}{ccccccccl} [E] & & & & [B] & & \neg C & \leftarrow & [D] \\ \neg E & \vee & A & \vee & \neg B & \vee & C & \leftarrow & \end{array}}{[A]} \quad \text{if } \neg E \ > \ A \ > \ \neg B \vee C$$

Backjump

$$\begin{array}{ccccccccl} & & \neg A \leftarrow & & [B] & & \neg C & \leftarrow & [D] \\ \neg E & \vee & A & \vee & \neg B & \vee & C & \leftarrow & \end{array}$$

Fail

Inference Rules by Example

Propagate

Positive

$$\frac{\begin{array}{lllll} & [B] & & \neg C & \leftarrow [D] \\ A \lor & \neg B & \lor & C & \leftarrow \end{array}}{A \leftarrow [B], [D]} \quad \text{if } A > \neg B \lor C$$

Decide

$$\frac{\begin{array}{lllllll} [E] & & & [B] & & \neg C & \leftarrow [D] \\ \neg E & \lor\ A & \lor & \neg B & \lor & C & \leftarrow \end{array}}{[A]} \quad \text{if } \neg E > A > \neg B \lor C$$

Backjump

Fail

$$\begin{array}{lllllll} [E] & & \neg A \leftarrow & [B] & & \neg C & \leftarrow [D] \\ \neg E & \lor & A \quad \lor & \neg B & \lor & C & \leftarrow \end{array}$$

Inference Rules by Example

Propagate

Positive

$$\frac{\begin{array}{cccl} & [B] & \neg C & \leftarrow [D] \\ A \lor & \neg B \lor & C & \leftarrow \end{array}}{A \leftarrow [B], [D]} \qquad \text{if } A > \neg B \lor C$$

Decide

$$\frac{\begin{array}{cccl} [E] & & [B] & \neg C \leftarrow [D] \\ \neg E \lor & A \lor & \neg B \lor & C \leftarrow \end{array}}{[A]} \qquad \text{if } \neg E > A > \neg B \lor C$$

Backjump

Fail

$$\frac{\begin{array}{cccl} [E] & \neg A \leftarrow & [B] & \neg C \leftarrow [D] \\ \neg E \lor & A \lor & \neg B \lor & C \leftarrow \end{array}}{\neg E \leftarrow [B], [D]}$$

Inference Rules by Example

Propagate

Positive

$$\frac{\begin{array}{cccccc} & & [B] & & \neg C & \leftarrow [D] \\ A & \lor & \neg B & \lor & C & \leftarrow \end{array}}{A \leftarrow [B], [D]} \qquad \text{if } A \; > \; \neg B \lor C$$

Decide

$$\frac{\begin{array}{cccccccc} [E] & & & & [B] & & \neg C & \leftarrow [D] \\ \neg E & \lor & A & \lor & \neg B & \lor & C & \leftarrow \end{array}}{[A]} \qquad \text{if } \neg E \; > \; A \; > \; \neg B \lor C$$

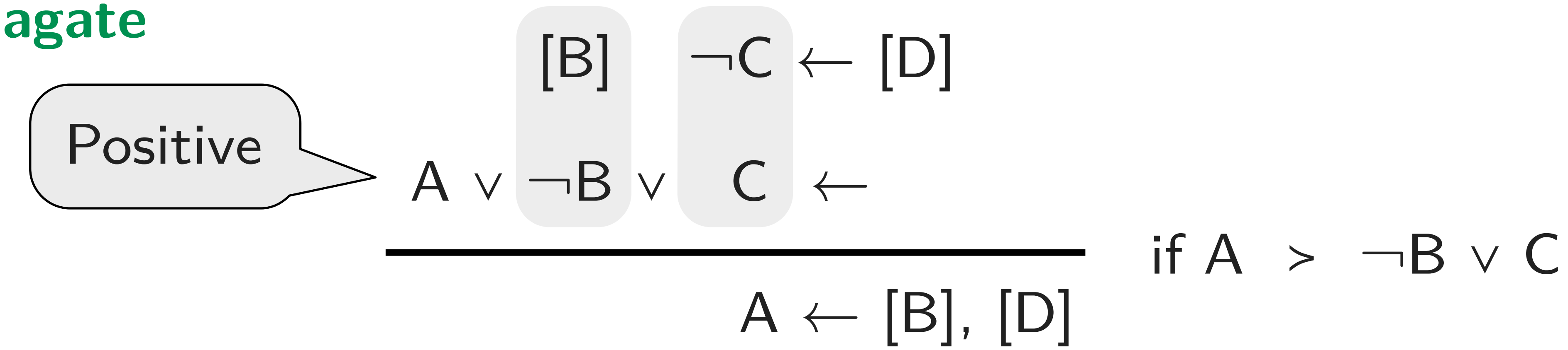
Backjump

Fail

$$\frac{\begin{array}{cccccccc} [E] & & \neg A \leftarrow & & [B] & & \neg C & \leftarrow [D] \\ \neg E & \lor & A & \lor & \neg B & \lor & C & \leftarrow \end{array}}{\neg E \leftarrow [B], [D]} \qquad \text{if } \neg E \; > \; A \lor \neg B \lor C$$

Inference Rules by Example

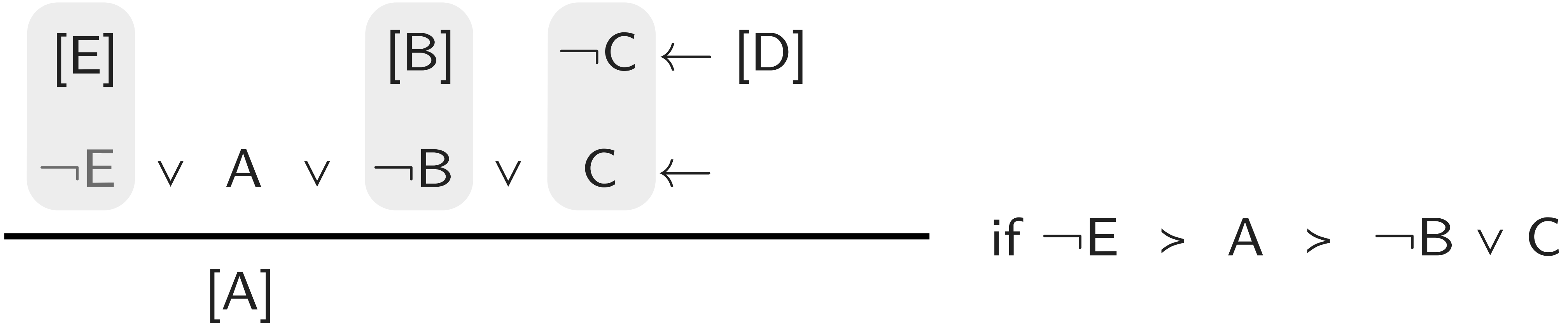
Propagate



Positive

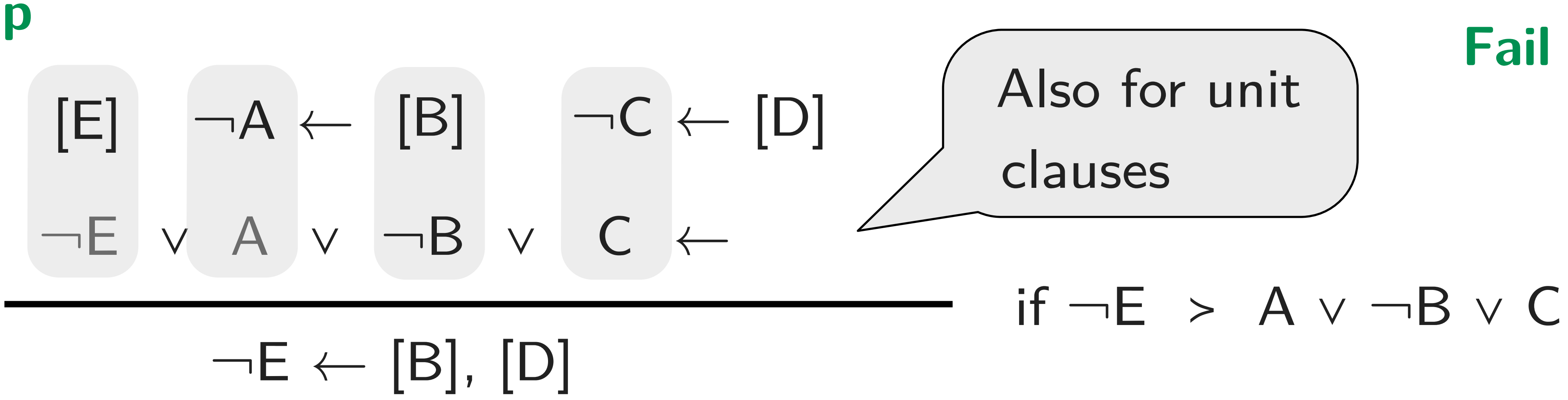
$$\frac{\begin{array}{ccccc} & [B] & & \neg C & \leftarrow [D] \\ A \lor & \neg B & \lor & C & \leftarrow \end{array}}{A \leftarrow [B], [D]} \quad \text{if } A > \neg B \lor C$$

Decide



$$\frac{\begin{array}{ccccccc} [E] & & & [B] & & \neg C & \leftarrow [D] \\ \neg E & \lor & A \lor & \neg B & \lor & C & \leftarrow \end{array}}{[A]} \quad \text{if } \neg E > A > \neg B \lor C$$

Backjump



Fail

Also for unit clauses

$$\frac{\begin{array}{ccccccc} [E] & & \neg A \leftarrow & [B] & & \neg C & \leftarrow [D] \\ \neg E & \lor & A & \lor\ \neg B & \lor & C & \leftarrow \end{array}}{\neg E \leftarrow [B], [D]} \quad \text{if } \neg E > A \lor \neg B \lor C$$

Inference Rules by Example

Propagate

Positive

$$\frac{\begin{array}{cccc} & [B] & \neg C \leftarrow [D] \\ A \lor & \neg B \lor & C \leftarrow \end{array}}{A \leftarrow [B], [D]} \quad \text{if } A > \neg B \lor C$$

Decide

$$\frac{\begin{array}{ccccc} [E] & & & [B] & \neg C \leftarrow [D] \\ \neg E \lor & A & \lor & \neg B \lor & C \leftarrow \end{array}}{[A]} \quad \text{if } \neg E > A > \neg B \lor C$$

Backjump

$$\frac{\begin{array}{cccc} [E] & \neg A \leftarrow & [B] & \neg C \leftarrow [D] \\ \neg E \lor & A \lor & \neg B \lor & C \leftarrow \end{array}}{\neg E \leftarrow [B], [D]} \quad \text{if } \neg E > A \lor \neg B \lor C$$

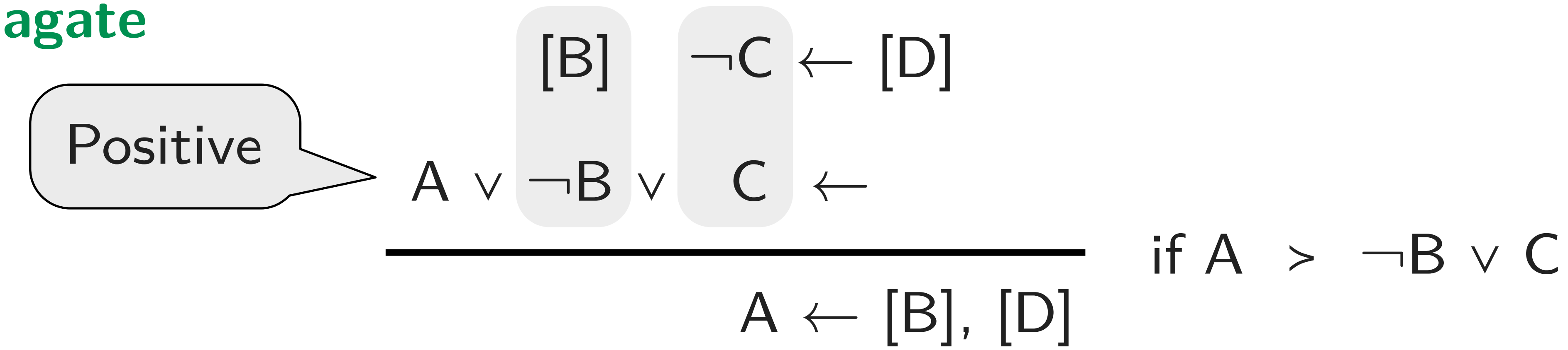
Also for unit clauses

Fail

...

Inference Rules by Example

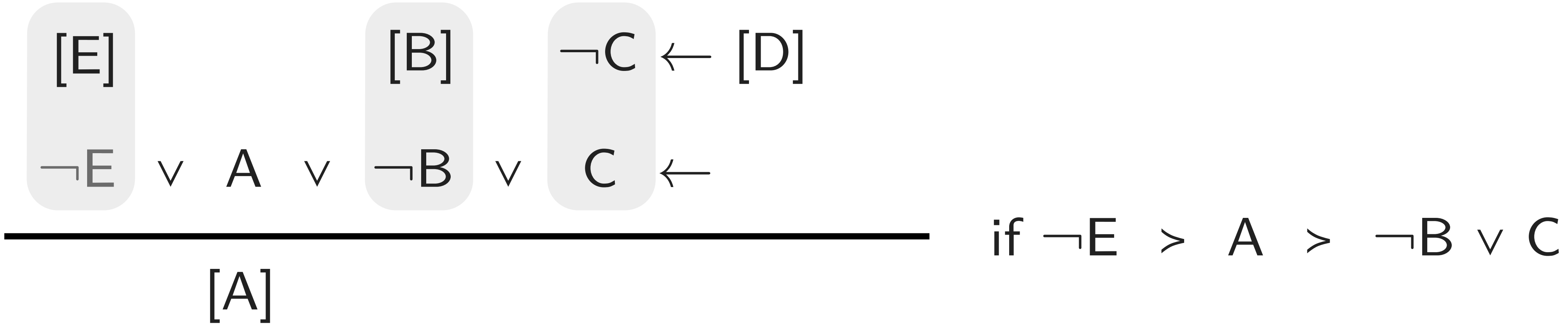
Propagate



Positive

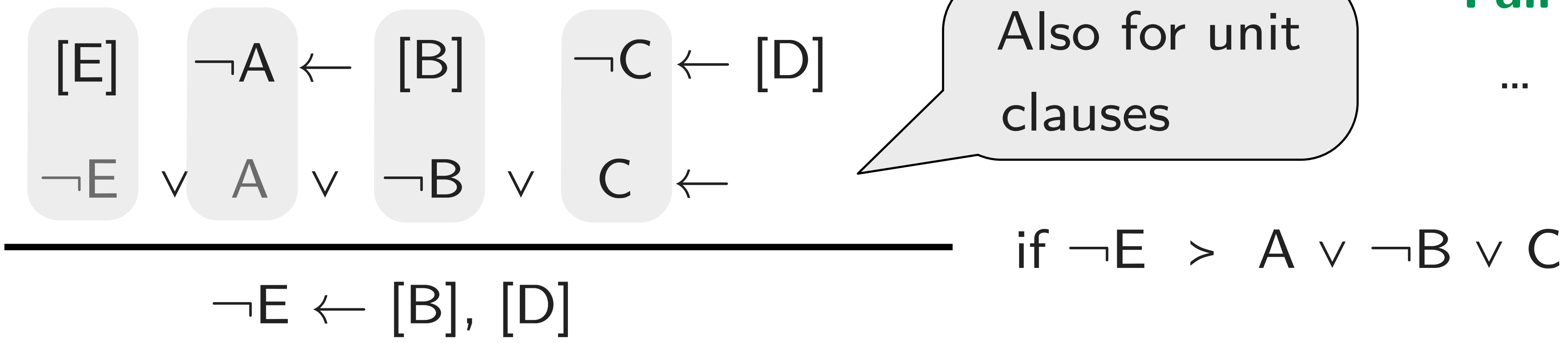
$$\frac{\begin{array}{llll} & [B] & \neg C \leftarrow [D] & \\ A \vee & \neg B \vee & C \leftarrow & \end{array}}{A \leftarrow [B], [D]} \quad \text{if } A > \neg B \vee C$$

Decide



$$\frac{\begin{array}{llll} [E] & & [B] & \neg C \leftarrow [D] \\ \neg E \vee & A \vee & \neg B \vee & C \leftarrow \end{array}}{[A]} \quad \text{if } \neg E > A > \neg B \vee C$$

Backjump



Also for unit clauses

$$\frac{\begin{array}{llll} [E] & \neg A \leftarrow & [B] & \neg C \leftarrow [D] \\ \neg E \vee & A \vee & \neg B \vee & C \leftarrow \end{array}}{\neg E \leftarrow [B], [D]} \quad \text{if } \neg E > A \vee \neg B \vee C$$

Fail

...

Invariant: clauses *head* ← *body* are ordered: min(*head*) > max(*body*)

Example CDCL-as-Saturation Derivation

$$F \vee D \leftarrow \qquad (0)$$

$$G \vee E \vee \neg D \leftarrow \qquad (1)$$

$$B \vee \neg A \leftarrow \qquad (2)$$

$$C \vee \neg B \leftarrow \qquad (3)$$

$$\neg D \vee \neg C \vee \neg A \leftarrow \qquad (4)$$

$$\neg C \vee A \leftarrow \qquad (5)$$

$$C \vee A \leftarrow \qquad (6)$$

$$D \vee \neg F \leftarrow \qquad (7)$$

Example CDCL-as-Saturation Derivation

$F \vee D \leftarrow$	(0)	$\neg D \vee \neg C \vee \neg A \leftarrow$	(4)
$G \vee E \vee \neg D \leftarrow$	(1)	$\neg C \vee A \leftarrow$	(5)
$B \vee \neg A \leftarrow$	(2)	$C \vee A \leftarrow$	(6)
$C \vee \neg B \leftarrow$	(3)	$D \vee \neg F \leftarrow$	(7)

[D] (Decide)

Example CDCL-as-Saturation Derivation

$F \vee D \leftarrow$ (0)

$G \vee E \vee \neg D \leftarrow$ (1)

$B \vee \neg A \leftarrow$ (2)

$C \vee \neg B \leftarrow$ (3)

$\neg D \vee \neg C \vee \neg A \leftarrow$ (4)

$\neg C \vee A \leftarrow$ (5)

$C \vee A \leftarrow$ (6)

$D \vee \neg F \leftarrow$ (7)

[D] (Decide)
[E] (Decide)

Example CDCL-as-Saturation Derivation

$F \vee D \leftarrow$ (0)

$G \vee E \vee \neg D \leftarrow$ (1)

$B \vee \neg A \leftarrow$ (2)

$C \vee \neg B \leftarrow$ (3)

$\neg D \vee \neg C \vee \neg A \leftarrow$ (4)

$\neg C \vee A \leftarrow$ (5)

$C \vee A \leftarrow$ (6)

$D \vee \neg F \leftarrow$ (7)

$[D]$	(Decide)
$[E]$	(Decide)
$[A]$	(Decide)

Example CDCL-as-Saturation Derivation

$F \vee D \leftarrow$ (0)

$G \vee E \vee \neg D \leftarrow$ (1)

$B \vee \neg A \leftarrow$ (2)

$C \vee \neg B \leftarrow$ (3)

$\neg D \vee \neg C \vee \neg A \leftarrow$ (4)

$\neg C \vee A \leftarrow$ (5)

$C \vee A \leftarrow$ (6)

$D \vee \neg F \leftarrow$ (7)

$[D]$	(Decide)
$[E]$	(Decide)
$[A]$	(Decide)
$B \leftarrow [A]$	(9 by 2,$[A]$)

Example CDCL-as-Saturation Derivation

$F \vee D \leftarrow$	(0)
$G \vee E \vee \neg D \leftarrow$	(1)
$B \vee \neg A \leftarrow$	(2)
$C \vee \neg B \leftarrow$	(3)
$\neg D \vee \neg C \vee \neg A \leftarrow$	(4)
$\neg C \vee A \leftarrow$	(5)
$C \vee A \leftarrow$	(6)
$D \vee \neg F \leftarrow$	(7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
$B \leftarrow [A]$	(9 by 2,[A])
$C \leftarrow [A]$	(10 by 3,9)

Example CDCL-as-Saturation Derivation

$F \lor D \leftarrow$ (0)

$G \lor E \lor \neg D \leftarrow$ (1)

$B \lor \neg A \leftarrow$ (2)

$C \lor \neg B \leftarrow$ (3)

$\neg D \lor \neg C \lor \neg A \leftarrow$ (4)

$\neg C \lor A \leftarrow$ (5)

$C \lor A \leftarrow$ (6)

$D \lor \neg F \leftarrow$ (7)

[D]	(Decide)
[E]	(Decide)
[A]	(Decide)
$B \leftarrow$ [A]	(9 by 2,[A])
$C \leftarrow$ [A]	(10 by 3,9)
	(conflict 4,10,[D],[A])

Example CDCL-as-Saturation Derivation

$F \lor D \leftarrow$ (0)

$G \lor E \lor \neg D \leftarrow$ (1)

$B \lor \neg A \leftarrow$ (2)

$C \lor \neg B \leftarrow$ (3)

$\neg D \lor \neg C \lor \neg A \leftarrow$ (4)

$\neg C \lor A \leftarrow$ (5)

$C \lor A \leftarrow$ (6)

$D \lor \neg F \leftarrow$ (7)

$[D]$	(Decide)
$[E]$	(Decide)
$[A]$	(Decide)
$B \leftarrow [A]$	(9 by 2,[A])
$C \leftarrow [A]$	(10 by 3,9)
	(conflict 4,10,[D],[A])
$\neg A \leftarrow [D]$	(11 by Backjump)

Example CDCL-as-Saturation Derivation

$F \lor D \leftarrow$ (0)

$G \lor E \lor \neg D \leftarrow$ (1)

$B \lor \neg A \leftarrow$ (2)

$C \lor \neg B \leftarrow$ (3)

$\neg D \lor \neg C \lor \neg A \leftarrow$ (4)

$\neg C \lor A \leftarrow$ (5)

$C \lor A \leftarrow$ (6)

$D \lor \neg F \leftarrow$ (7)

$[D]$	(Decide)
$[E]$	(Decide)
$[A]$	(Decide)
$B \leftarrow [A]$	(9 by 2,[A])
$C \leftarrow [A]$	(10 by 3,9)
	(conflict 4,10,[D],[A])
$\neg A \leftarrow [D]$	(11 by Backjump)

[A] overridden by $\neg A \leftarrow [D]$ and [D]

Example CDCL-as-Saturation Derivation

$\mathsf{F} \vee \mathsf{D} \leftarrow$ (0)

$\mathsf{G} \vee \mathsf{E} \vee \neg\mathsf{D} \leftarrow$ (1)

$\mathsf{B} \vee \neg\mathsf{A} \leftarrow$ (2)

$\mathsf{C} \vee \neg\mathsf{B} \leftarrow$ (3)

$\neg\mathsf{D} \vee \neg\mathsf{C} \vee \neg\mathsf{A} \leftarrow$ (4)

$\neg\mathsf{C} \vee \mathsf{A} \leftarrow$ (5)

$\mathsf{C} \vee \mathsf{A} \leftarrow$ (6)

$\mathsf{D} \vee \neg\mathsf{F} \leftarrow$ (7)

$[\mathsf{D}]$ (Decide)

$[\mathsf{E}]$ (Decide)

$[\mathsf{A}]$ (Decide)

$\mathsf{B} \leftarrow [\mathsf{A}]$ (9 by 2,[A])

$\mathsf{C} \leftarrow [\mathsf{A}]$ (10 by 3,9)

(conflict 4,10,[D],[A])

$\neg\mathsf{A} \leftarrow [\mathsf{D}]$ (11 by Backjump)

$\mathsf{C} \leftarrow [\mathsf{D}]$ (12 by 6,11)

[A] overridden by $\neg\mathsf{A} \leftarrow [\mathsf{D}]$ and [D]

Example CDCL-as-Saturation Derivation

$F \lor D \leftarrow$ (0)

$G \lor E \lor \neg D \leftarrow$ (1)

$B \lor \neg A \leftarrow$ (2)

$C \lor \neg B \leftarrow$ (3)

$\neg D \lor \neg C \lor \neg A \leftarrow$ (4)

$\neg C \lor A \leftarrow$ (5)

$C \lor A \leftarrow$ (6)

$D \lor \neg F \leftarrow$ (7)

$[D]$ (Decide)

$[E]$ (Decide)

$[A]$ (Decide)

$B \leftarrow [A]$ (9 by 2,[A])

$C \leftarrow [A]$ (10 by 3,9)

(conflict 4,10,[D],[A])

$\neg A \leftarrow [D]$ (11 by Backjump)

$C \leftarrow [D]$ (12 by 6,11)

(conflict 5,11,12)

$[A]$ overridden by $\neg A \leftarrow [D]$ and $[D]$

Example CDCL-as-Saturation Derivation

$F \vee D \leftarrow$ (0)
$G \vee E \vee \neg D \leftarrow$ (1)
$B \vee \neg A \leftarrow$ (2)
$C \vee \neg B \leftarrow$ (3)
$\neg D \vee \neg C \vee \neg A \leftarrow$ (4)
$\neg C \vee A \leftarrow$ (5)
$C \vee A \leftarrow$ (6)
$D \vee \neg F \leftarrow$ (7)

$[D]$ (Decide)
$[E]$ (Decide)
$[A]$ (Decide)
$B \leftarrow [A]$ (9 by 2,[A])
$C \leftarrow [A]$ (10 by 3,9)
(conflict 4,10,[D],[A])
$\neg A \leftarrow [D]$ (11 by Backjump)

$C \leftarrow [D]$ (12 by 6,11)
(conflict 5,11,12)
$\neg D \leftarrow$ (13 by Backjump)

[A] overridden by $\neg A \leftarrow [D]$ and [D]

Example CDCL-as-Saturation Derivation

$F \vee D \leftarrow$	(0)	$\neg D \vee \neg C \vee \neg A \leftarrow$	(4)
$G \vee E \vee \neg D \leftarrow$	(1)	$\neg C \vee A \leftarrow$	(5)
$B \vee \neg A \leftarrow$	(2)	$C \vee A \leftarrow$	(6)
$C \vee \neg B \leftarrow$	(3)	$D \vee \neg F \leftarrow$	(7)

$[D]$	(Decide)
$[E]$	(Decide)
$[A]$	(Decide)
$B \leftarrow [A]$	(9 by 2,[A])
$C \leftarrow [A]$	(10 by 3,9)
	(conflict 4,10,[D],[A])
$\neg A \leftarrow [D]$	(11 by Backjump)

$C \leftarrow [D]$	(12 by 6,11)
	(conflict 5,11,12)
$\neg D \leftarrow$	(13 by Backjump)

[D] overridden by $\neg D \leftarrow$

[A] no longer overridden by $\neg A \leftarrow [D]$ and [D]

[A] overridden by $\neg A \leftarrow [D]$ and [D]

Example CDCL-as-Saturation Derivation

$F \vee D \leftarrow$ (0)

$G \vee E \vee \neg D \leftarrow$ (1)

$B \vee \neg A \leftarrow$ (2)

$C \vee \neg B \leftarrow$ (3)

$\neg D \vee \neg C \vee \neg A \leftarrow$ (4)

$\neg C \vee A \leftarrow$ (5)

$C \vee A \leftarrow$ (6)

$D \vee \neg F \leftarrow$ (7)

$[D]$ (Decide)

$[E]$ (Decide)

$[A]$ (Decide)

$B \leftarrow [A]$ (9 by 2,[A])

$C \leftarrow [A]$ (10 by 3,9)

(conflict 4,10,[D],[A])

$\neg A \leftarrow [D]$ (11 by Backjump)

[A] overridden by $\neg A \leftarrow [D]$ and [D]

$C \leftarrow [D]$ (12 by 6,11)

(conflict 5,11,12)

$\neg D \leftarrow$ (13 by Backjump)

[D] overridden by $\neg D \leftarrow$

[A] no longer overridden

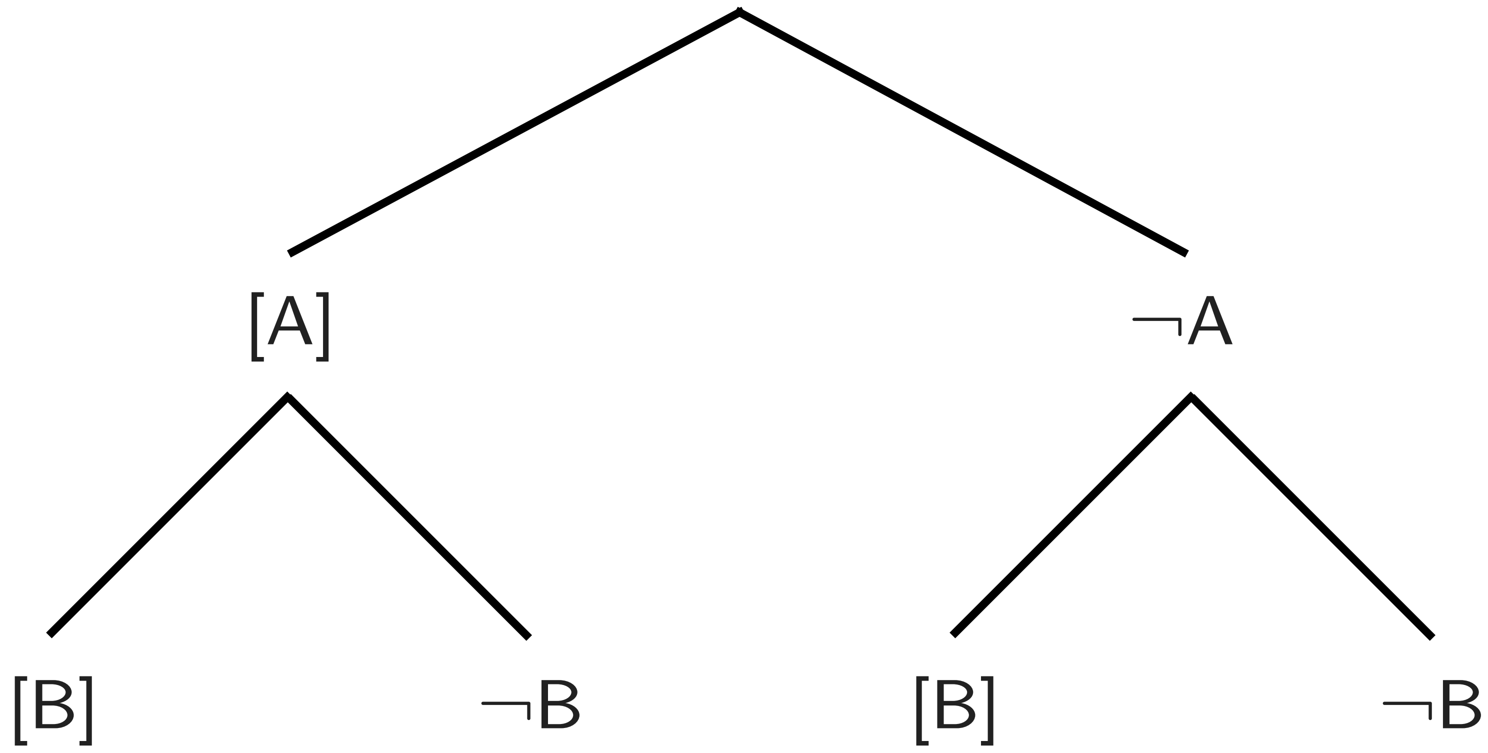
by $\neg A \leftarrow [D]$ and [D]

$F \leftarrow$ (14 by 0,13)

$\square \leftarrow$ (FAIL 7)

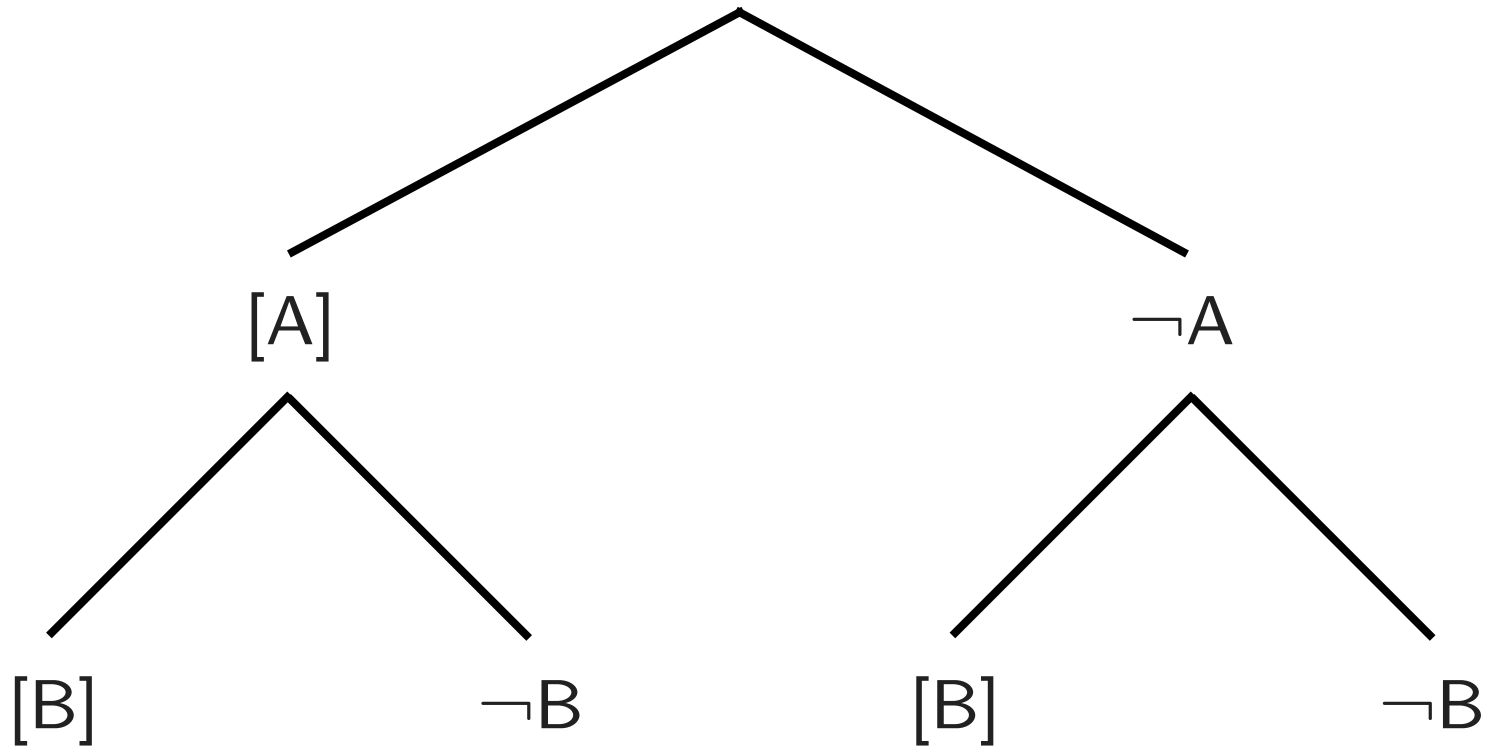
Overriding

Switching interpretations by increasing clause sets



Overriding

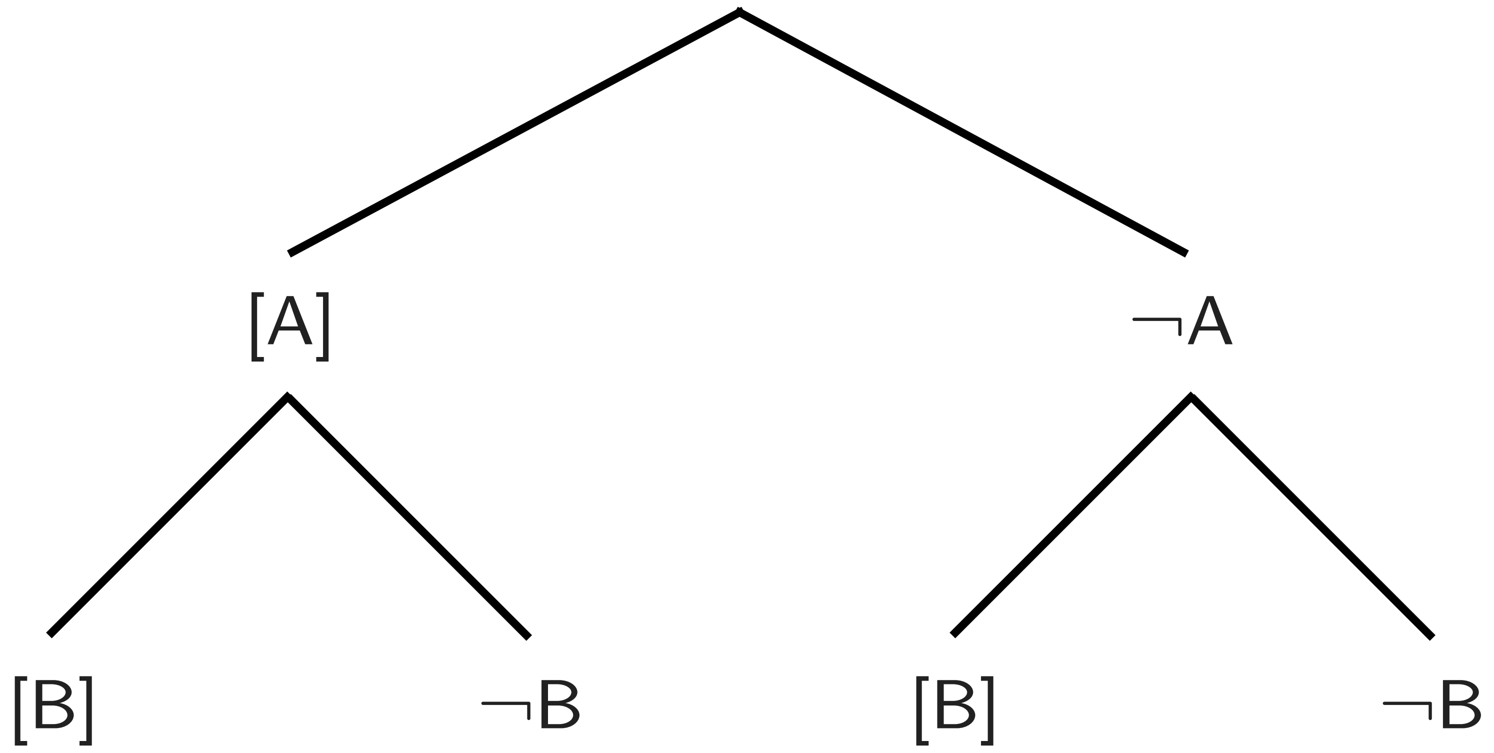
Switching interpretations by increasing clause sets



[A]

Overriding

Switching interpretations by increasing clause sets

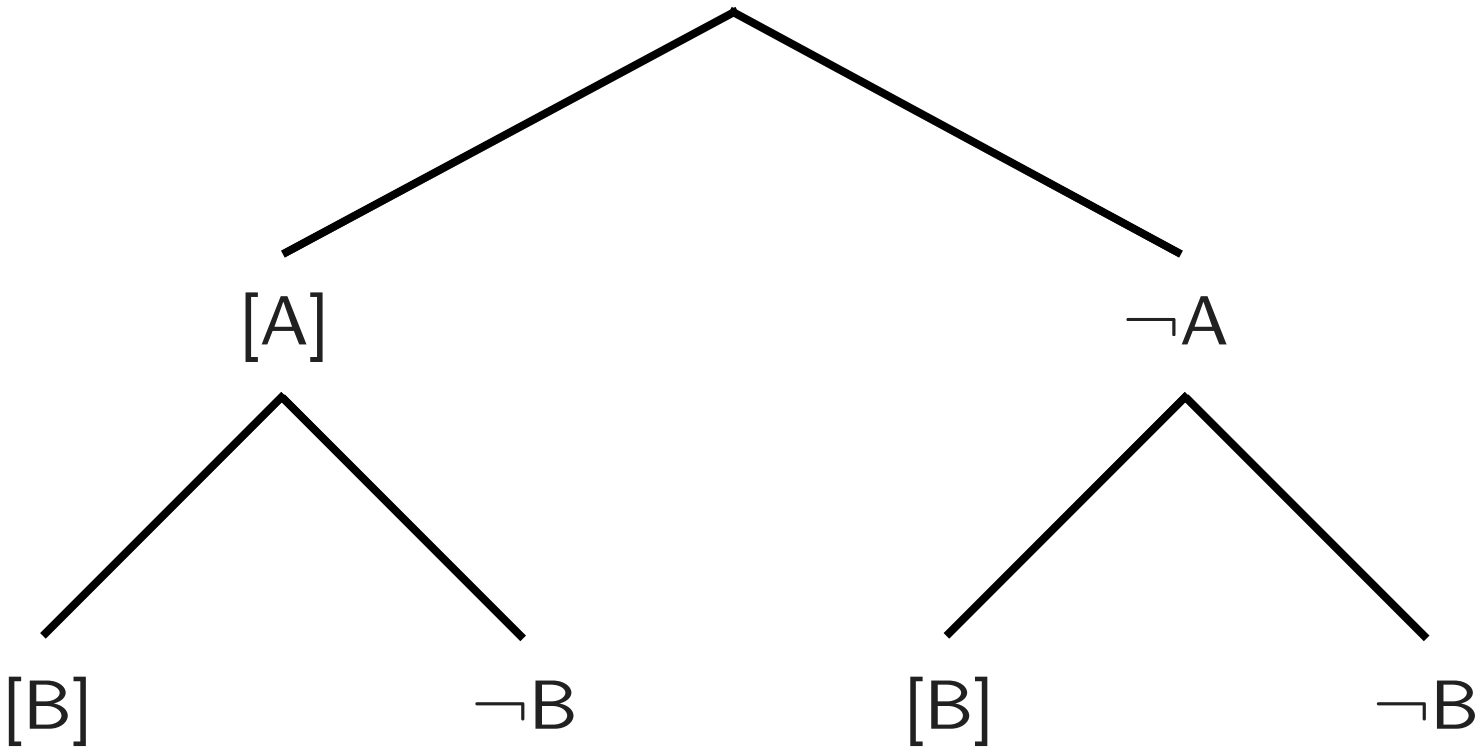


[A]

[B]

Overriding

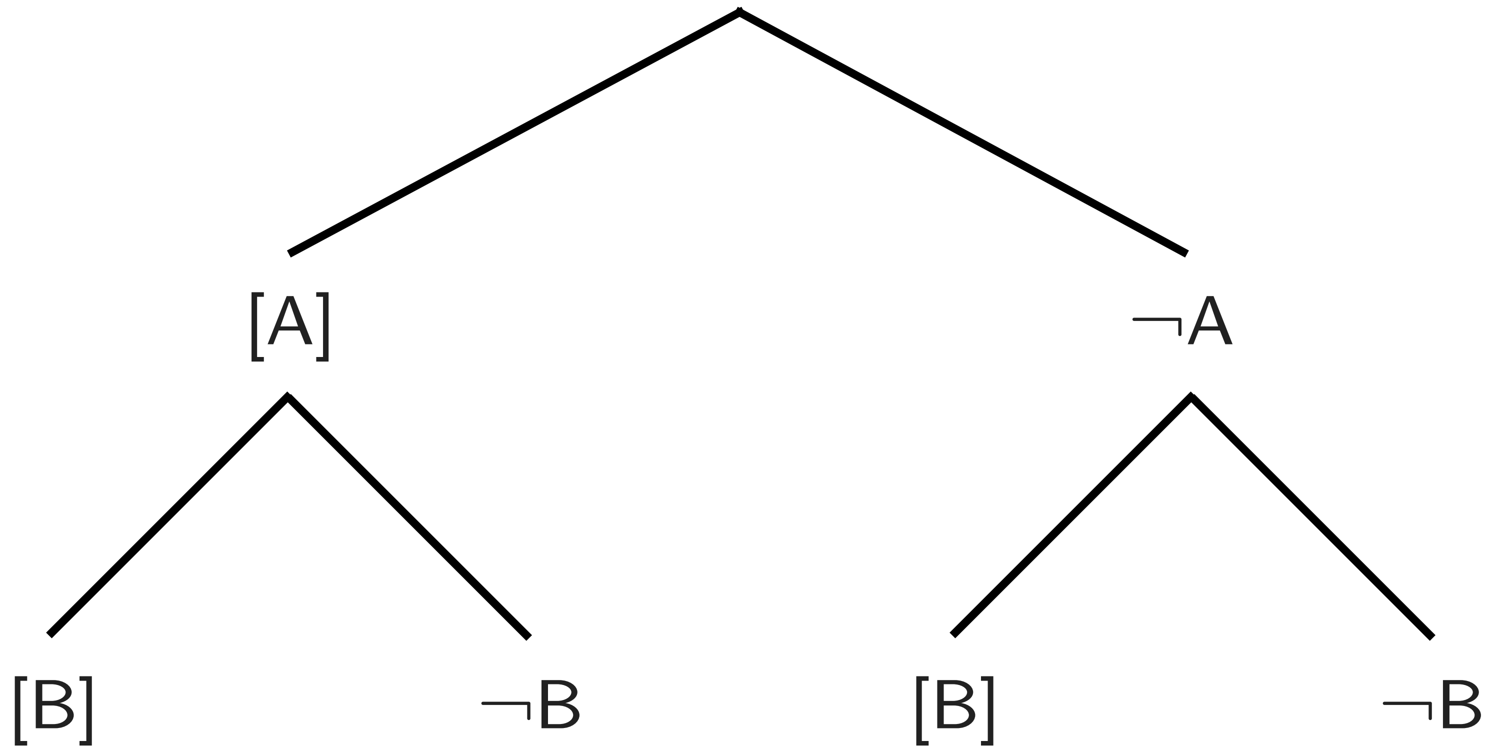
Switching interpretations by increasing clause sets



[A]	[A]
[B]	[B]

Overriding

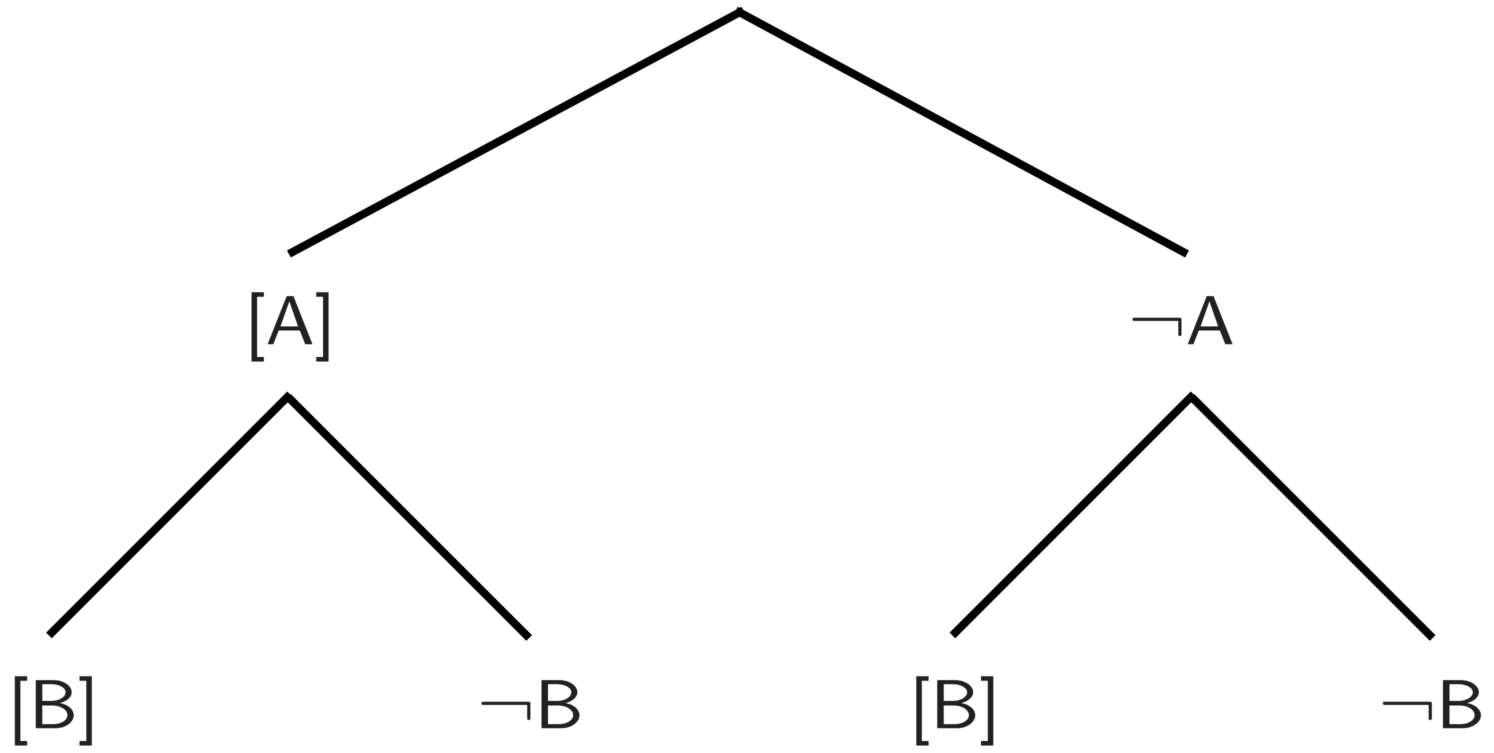
Switching interpretations by increasing clause sets



[A]	[A]
[B]	[B]
	$\neg B \leftarrow [A]$

Overriding

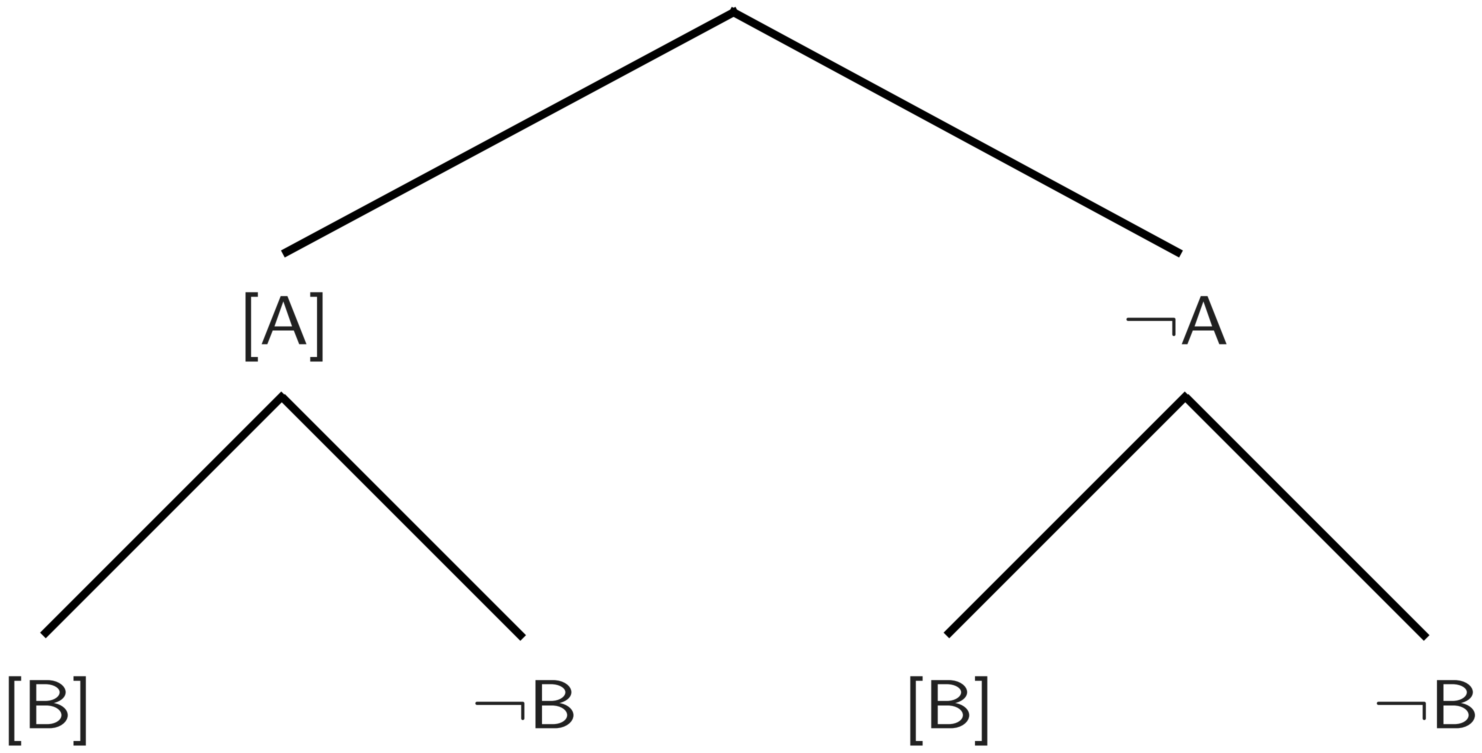
Switching interpretations by increasing clause sets



[A]	[A]	[A]	
[B]	[B]	[B]	
	¬B ← [A]	¬B ← [A]	

Overriding

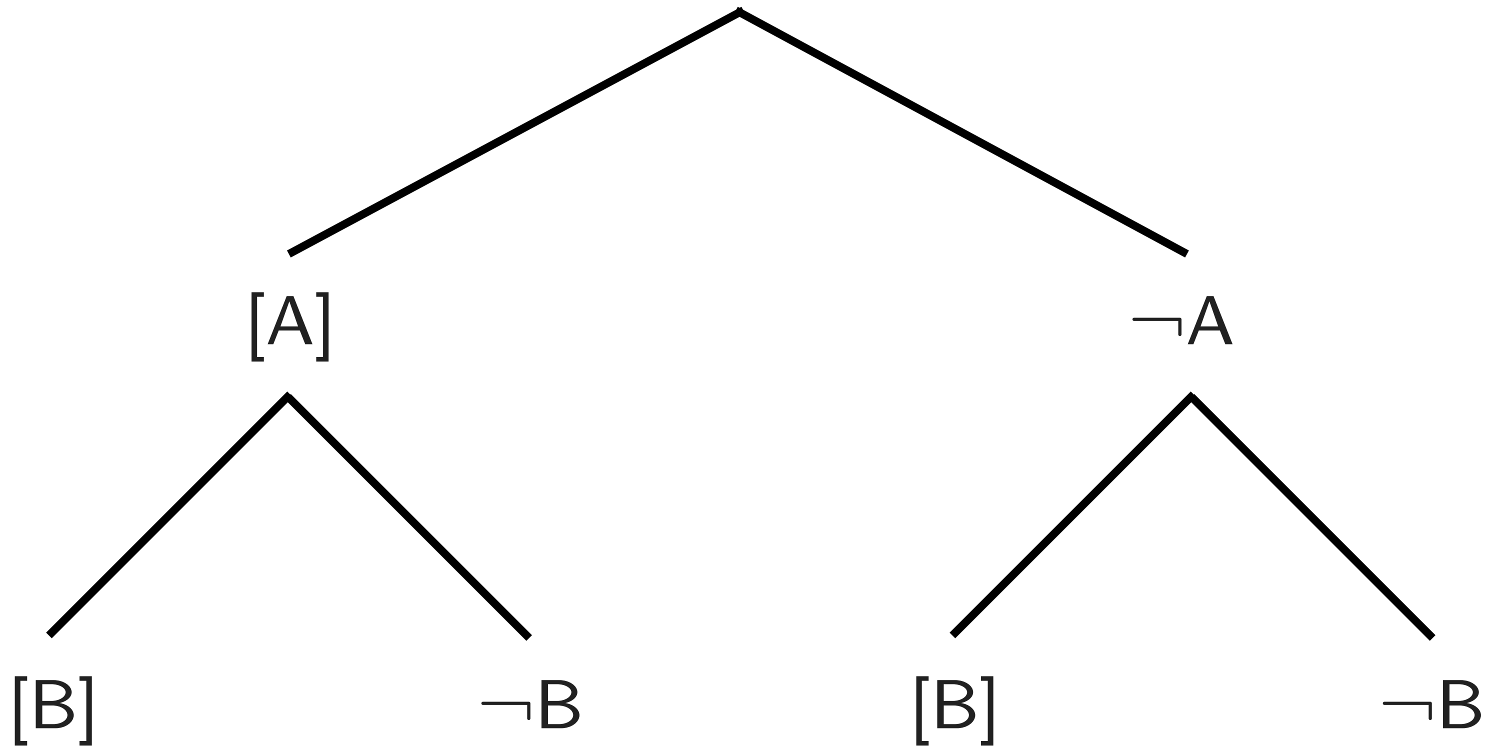
Switching interpretations by increasing clause sets



[A]	[A]	[A]
[B]	[B]	[B]
	$\neg B \leftarrow [A]$	$\neg B \leftarrow [A]$
		$\neg A \leftarrow$

Overriding

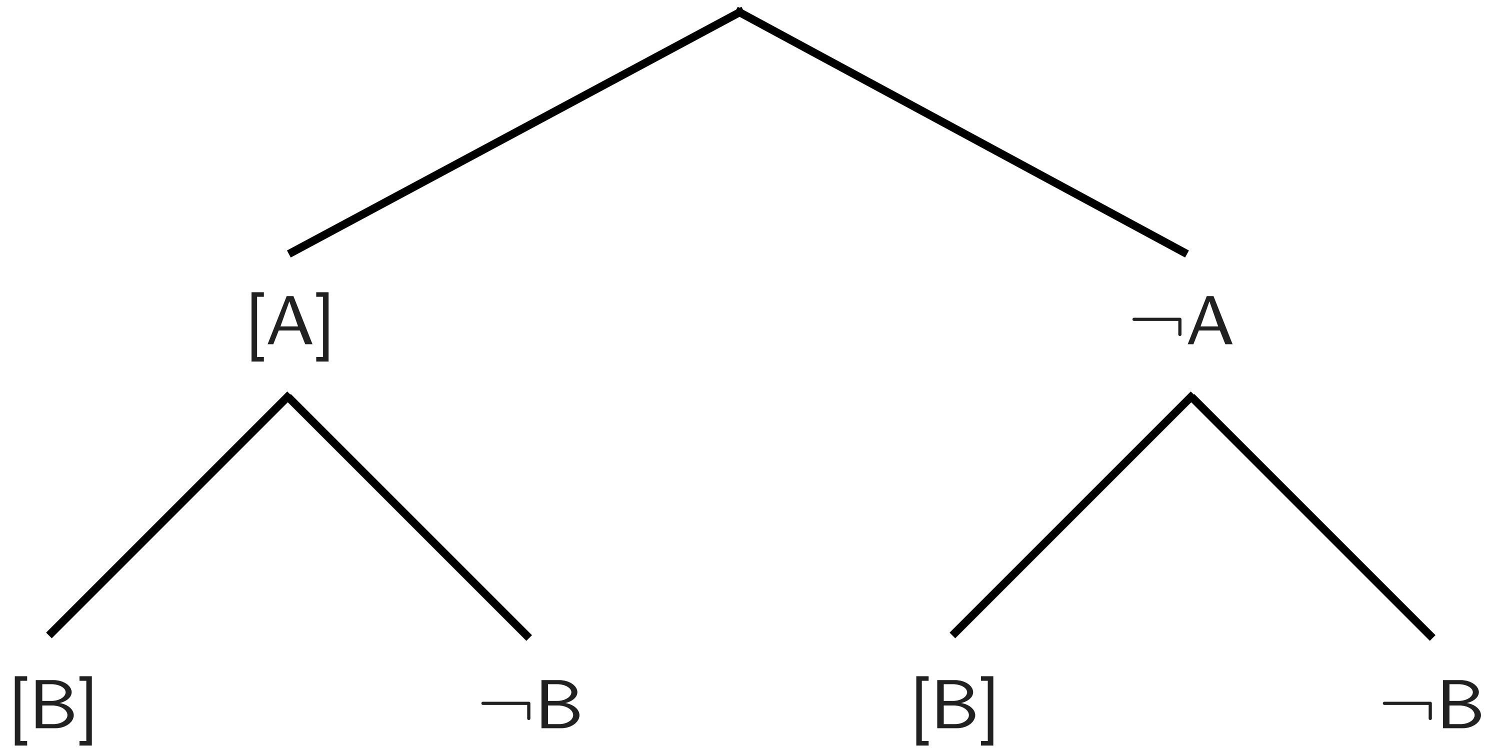
Switching interpretations by increasing clause sets



[A]	[A]	[A]	[A]
[B]	[B]	[B]	[B]
	$\neg B \leftarrow [A]$	$\neg B \leftarrow [A]$	$\neg B \leftarrow [A]$
		$\neg A \leftarrow$	$\neg A \leftarrow$

Overriding

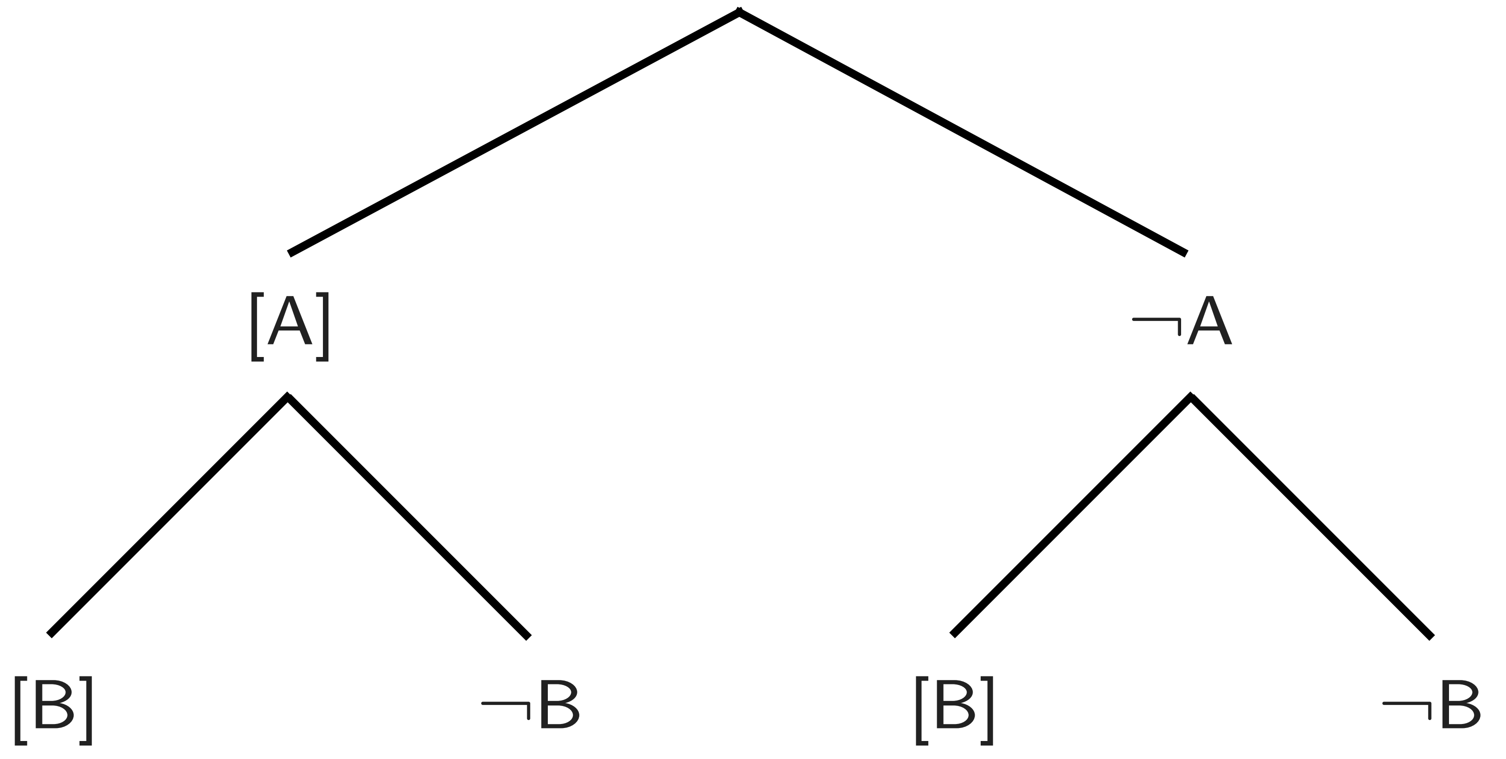
Switching interpretations by increasing clause sets



$[A]$	$[A]$	$[A]$	$[A]$
$[B]$	$[B]$	$[B]$	$[B]$
	$\neg B \leftarrow [A]$	$\neg B \leftarrow [A]$	$\neg B \leftarrow [A]$
		$\neg A \leftarrow$	$\neg A \leftarrow$
			$\neg B \leftarrow$

Overriding

Switching interpretations by increasing clause sets



$[A]$ $\neg A$

$[B]$ $\neg B$ $[B]$ $\neg B$

$[A]$	$[A]$	$[A]$	$[A]$
$[B]$	$[B]$	$[B]$	$[B]$
	$\neg B \leftarrow [A]$	$\neg B \leftarrow [A]$	$\neg B \leftarrow [A]$
		$\neg A \leftarrow$	$\neg A \leftarrow$
			$\neg B \leftarrow$

Clauses are ordered

$\Rightarrow$ overriding is well-founded

Redundant Decision Literals and Clauses

Locally redundant ≈ redundancy dependent on decision literal context

Global redundant ≈ redundancy independent of decision literal context

Let $A > B > C > D > E \;\Rightarrow\; (9) > (8) > \cdots > (1)$

$$[E] \quad (1)$$
$$\neg D \leftarrow [E] \quad (2)$$
$$[D] \quad (3)$$
$$[C] \quad (4)$$
$$[B] \quad (5)$$
$$A \leftarrow [D] \quad (6)$$
$$A \leftarrow [D], [E] \quad (7)$$
$$A \leftarrow [C], [E] \quad (8)$$
$$A \leftarrow [B] \quad (9)$$

Redundant Decision Literals and Clauses

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Let $A > B > C > D > E \ \Rightarrow\ (9) > (8) > \cdots > (1)$

$$[E] \tag{1}$$

$$\neg D \leftarrow [E] \tag{2}$$

$$[D] \tag{3}$$

$$[C] \tag{4}$$

$$[B] \tag{5}$$

$$A \leftarrow [D] \tag{6}$$

$$A \leftarrow [D], [E] \tag{7}$$

$$A \leftarrow [C], [E] \tag{8}$$

$$A \leftarrow [B] \tag{9}$$

(3) locally redundant, as overridden by (2)

Redundant Decision Literals and Clauses

Locally redundant ≈ redundancy dependent on decision literal context

Global redundant ≈ redundancy independent of decision literal context

Let $A > B > C > D > E \;\Rightarrow\; (9) > (8) > \cdots > (1)$

$$\begin{array}{ll}
[E] & (1) \\
\neg D \leftarrow [E] & (2) \\
[D] & (3) \\
[C] & (4) \\
[B] & (5) \\
A \leftarrow [D] & (6) \\
A \leftarrow [D], [E] & (7) \\
A \leftarrow [C], [E] & (8) \\
A \leftarrow [B] & (9)
\end{array}$$

(3) locally redundant, as overridden by (2)

(6) locally redundant, as [D] locally redundant

Redundant Decision Literals and Clauses

Locally redundant ≈ redundancy dependent on decision literal context

Global redundant ≈ redundancy independent of decision literal context

Let $A > B > C > D > E \;\Rightarrow\; (9) > (8) > \cdots > (1)$

$[E]$ (1)

$\neg D \leftarrow [E]$ (2)

$[D]$ (3) — (3) locally redundant, as overridden by (2)

$[C]$ (4)

$[B]$ (5)

$A \leftarrow [D]$ (6) — (6) locally redundant, as [D] locally redundant

$A \leftarrow [D], [E]$ (7) — (7) globally redundant, as subsumed by (6)

$A \leftarrow [C], [E]$ (8)

$A \leftarrow [B]$ (9)

Redundant Decision Literals and Clauses

Locally redundant ≈ redundancy dependent on decision literal context

Global redundant ≈ redundancy independent of decision literal context

Let $A > B > C > D > E \Rightarrow (9) > (8) > \cdots > (1)$

$$\begin{array}{ll}
[E] & (1) \\
\neg D \leftarrow [E] & (2) \\
[D] & (3) \\
[C] & (4) \\
[B] & (5) \\
A \leftarrow [D] & (6) \\
A \leftarrow [D], [E] & (7) \\
A \leftarrow [C], [E] & (8) \\
A \leftarrow [B] & (9)
\end{array}$$

(3) locally redundant, as overridden by (2)

(6) locally redundant, as [D] locally redundant

(7) globally redundant, as subsumed by (6)

(9) locally redundant by (8)

Redundant Decision Literals and Clauses

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Global redundant ≈ redundancy independent of decision literal context

Let $A > B > C > D > E \;\Rightarrow\; (9) > (8) > \cdots > (1)$

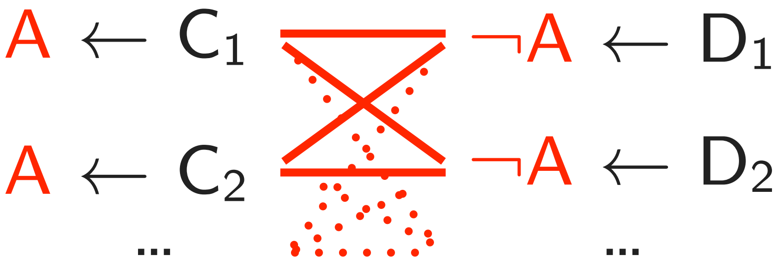
$[E]$	(1)	
$\neg D \leftarrow [E]$	(2)	
$[D]$	(3)	(3) locally redundant, as overridden by (2)
$[C]$	(4)	
$[B]$	(5)	
$A \leftarrow [D]$	(6)	(6) locally redundant, as [D] locally redundant
$A \leftarrow [D], [E]$	(7)	(7) globally redundant, as subsumed by (6)
$A \leftarrow [C], [E]$	(8)	
$A \leftarrow [B]$	(9)	(9) locally redundant by (8)

⇒ at any time ≤ 1 locally non-redundant unit clauses with same head (here A)

Redundant Inferences

Inferences from locally redundant clauses can be deferred

⇒ avoids clause recombination problem, e.g., for Backjump:



$A \leftarrow C_1$ $\qquad$ $\neg A \leftarrow D_1$

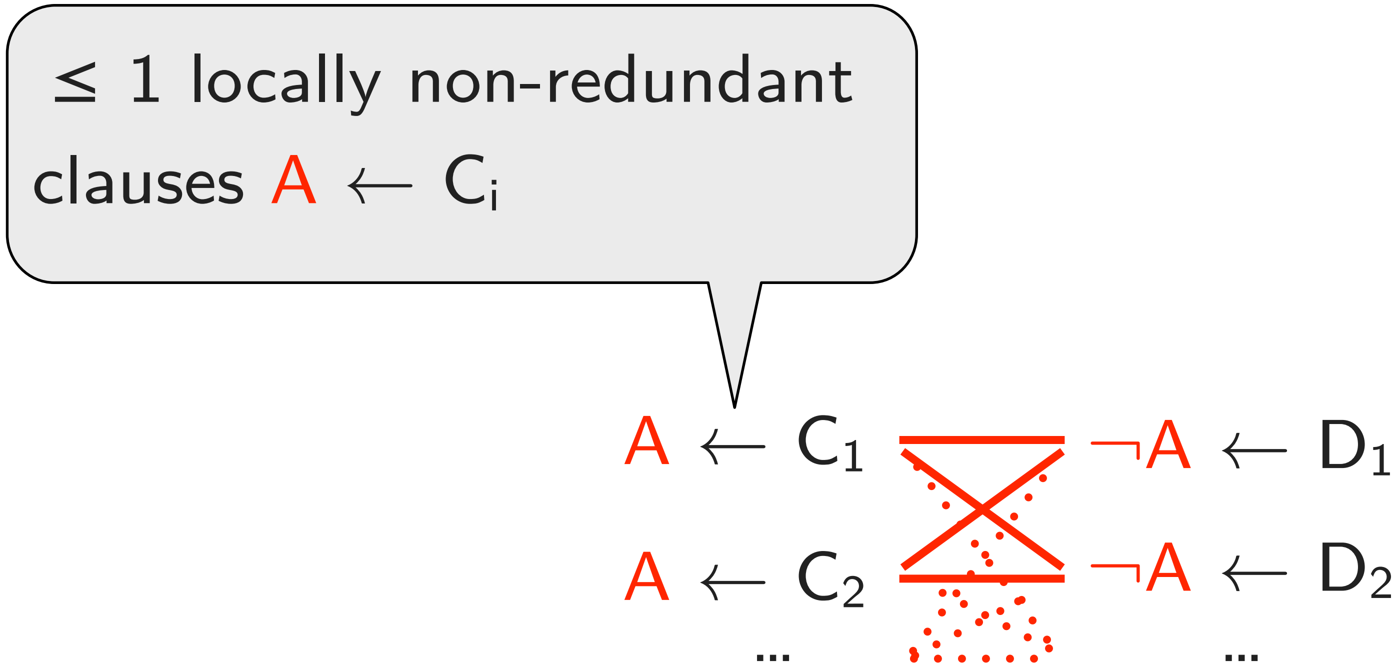
$A \leftarrow C_2$ $\qquad$ $\neg A \leftarrow D_2$

... $\qquad$ ...

Redundant Inferences

Inferences from locally redundant clauses can be deferred

⇒ avoids clause recombination problem, e.g., for Backjump:



$\leq$ 1 locally non-redundant clauses $A \leftarrow C_i$

$A \leftarrow C_1$ $\quad$ $\neg A \leftarrow D_1$

$A \leftarrow C_2$ $\quad$ $\neg A \leftarrow D_2$

... $\quad$ ...

Redundant Inferences

Inferences from locally redundant clauses can be deferred

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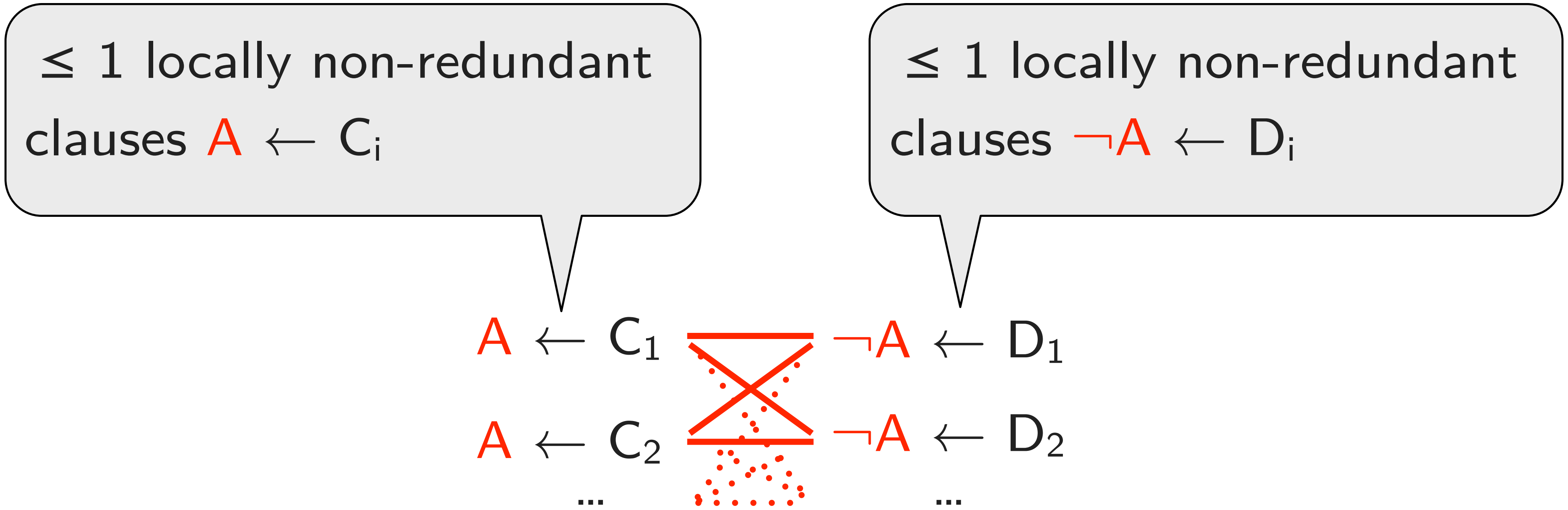

≤ 1 locally non-redundant clauses $A \leftarrow C_i$

≤ 1 locally non-redundant clauses $\neg A \leftarrow D_i$

$A \leftarrow C_1$ $\neg A \leftarrow D_1$

$A \leftarrow C_2$ $\neg A \leftarrow D_2$

... ...

Redundant Inferences

Inferences from locally redundant clauses can be deferred

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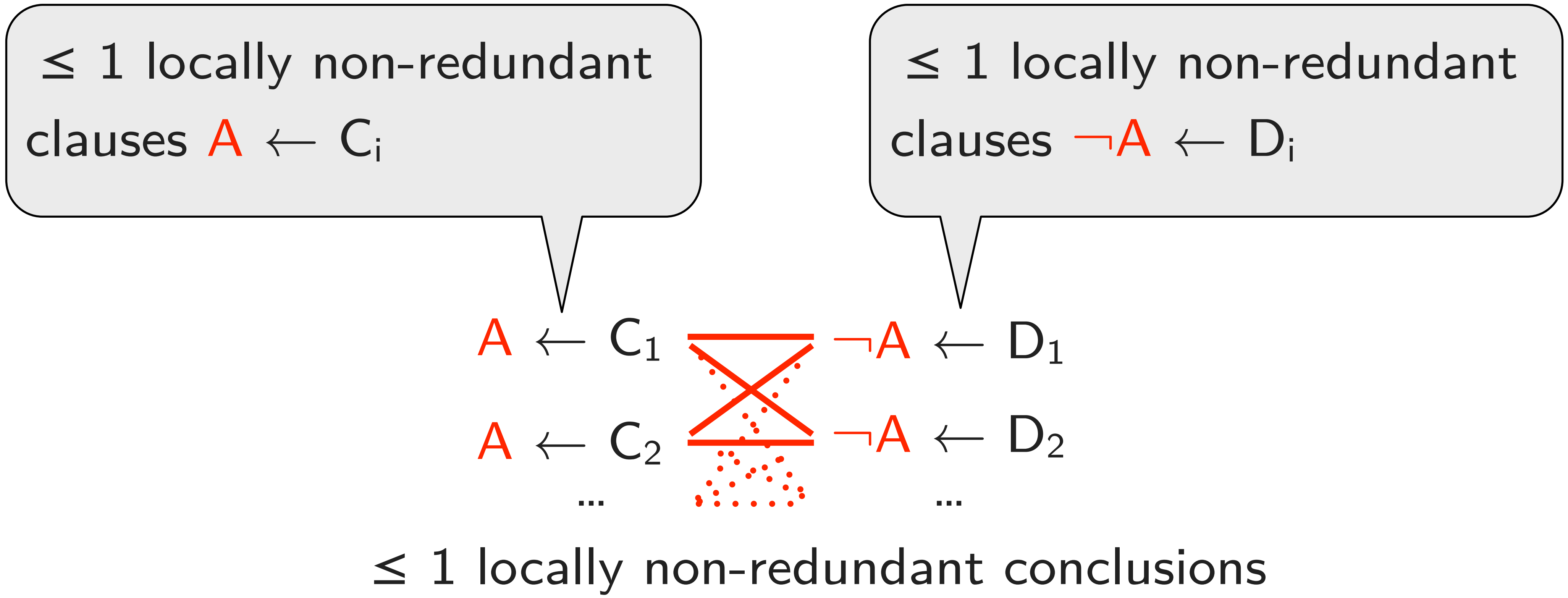


≤ 1 locally non-redundant clauses $A \leftarrow C_i$

≤ 1 locally non-redundant clauses $\neg A \leftarrow D_i$

$A \leftarrow C_1$

$A \leftarrow C_2$

...

$\neg A \leftarrow D_1$

$\neg A \leftarrow D_2$

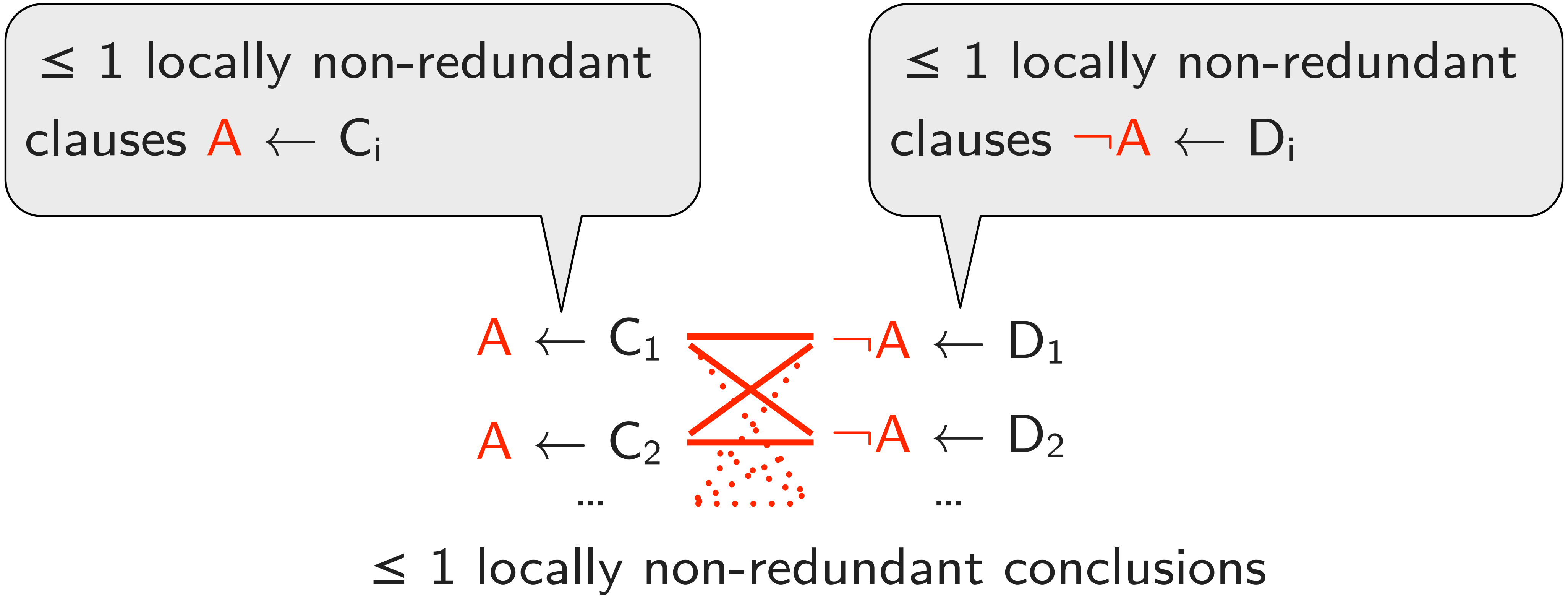
...

≤ 1 locally non-redundant conclusions

Redundant Inferences

Inferences from locally redundant clauses can be deferred

⇒ avoids clause recombination problem, e.g., for Backjump:



≤ 1 locally non-redundant clauses $A \leftarrow C_i$

≤ 1 locally non-redundant clauses $\neg A \leftarrow D_i$

$A \leftarrow C_1$ $\neg A \leftarrow D_1$

$A \leftarrow C_2$ $\neg A \leftarrow D_2$

... ...

≤ 1 locally non-redundant conclusions

Similarly for Propagate

Model Construction $I(M \blacksquare N)$

Let $M \blacksquare N$ be a state

Let $e \in M \cup N$ be the next expression considered

Suppose set of literals J_e has been defined for all f with $e > f$

Extend J_e as follows

Case	$e = A \leftarrow D$	$<$	$e = \neg A \leftarrow D$	$<$	$e = [A]$
Result	$J_e \cup \{A\}$ if (1) $D \subseteq M_N$ (2) $A \notin J_e$		$J_e \cup \{\neg A\}$ if (1) $D \subseteq M_N$ (2) $\neg A \notin J_e$ (3) $A \notin J_e$		$J_e \cup \{A\}$ if (1) $A \notin J_e$ (2) $\neg A \notin J_e$

where $M_N = \{\ [A] \in M \mid [A]$ is not locally redundant wrt. $M \blacksquare N\ \}$

Define $I(M \blacksquare N)$ as the interpretation obtained from the final set J

Completeness

Inference is **locally redundant** ≈ some premise or conclusion is locally redundant

State $M \blacksquare N$ is **saturated** iff every inference from $M \blacksquare N$ is locally redundant

Satisfaction relation

$$(M, I) \models C \leftarrow D \quad \text{iff} \quad D \not\subseteq M \text{ or } I \models C$$

Theorem (static completeness)

Let $M \blacksquare N$ be a saturated state such that for all $C \leftarrow D \in N$, $|C| = 1$ or $D = \{\}$ (i.e., unit or ordinary clauses only). If $\square \leftarrow \notin N$ then $(M_N, I(M \blacksquare N)) \models N$.

Dynamic completeness result with simplification (straightforward?)

First-Order Logic

Lifting

Local redundancy: semantics is straightforward, via ground instances

$[Q(x)]$	$Q(a)$, $Q(b)$, ...
$[P(x)]$	$P(b)$, ...
$\neg P(a) \leftarrow [Q(a)]$	

Inference rules: straightforward

$P(x) \vee Q(x) \leftarrow$	(1)
$\neg Q(a) \leftarrow$	(2)
$P(a) \leftarrow$	(3 by 1,2)
$[Q(x)] \leftarrow$	(Decide)

Conclusion: Issues

Differences to CDCL

Backjumping

- Does not remove clauses, only makes them locally redundant

Use of ordering

- In general need Decide even in Horn case
- Fix: construct/modify ordering on-the-fly (compatible with *local* redundancy)

Local redundancy

How to compute l.r. effectively/efficiently?

- Watch 1 clause per decision literal [A] that overrides [A], if any

However, for first-order logic:

- l.r. is optional - graceful degradation with decreasing precision
- l.r. acts as interface to model representation:
- can use whatever suits best, e.g., contexts, DIGs, constraint literals,...