Splitting an operator

An algebraic modularity result and its application to logic programming

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Logic Programming
↓
Abstraction Theory
↓
Stratification

Various Logic Program Semantics

- Assign "meaning" to a program / knowledge base: perfect model, stable models, **well-founded model**
- Normal (logic) programs: negation in rule body allowed.

$$win(X) \leftarrow move(X, Y),\ not\ win(Y) \quad (1)$$
$$move(c, d) \leftarrow \quad (2)$$
$$move(a, b) \leftarrow \quad (3)$$
$$move(b, a) \leftarrow \quad (4)$$

- **The** well-founded model:

True	Undefined	False
$win(c)$	$win(a)$	$win(d)$
	$win(b)$	

- **Two** stable models:

(i)

True	False
$win(c)$	$win(d)$
$win(a)$	$win(b)$

(ii)

True	False
$win(c)$	$win(d)$
$win(b)$	$win(a)$

More About Well-Founded Models

- See [VanGelder/Ross/Schlipf 89, Przymusinski 91]
- Generally accepted for "reasonable" sceptical reasoning
- "well-behaved" :
  - always exists, stratification not required
  - unique model
  - goal-oriented procedure exists
  - quadratic complexity
- *undef* is assigned to atoms which negatively depend on themselves, and for which no independent "well-founded" derivation exists
- XSB-Prolog system (Warren et. al., top-down system)
- SModels (Niemelä et. al., bottom-up system, also for stable model semantics)

"Building in" Information into Programs

- Program P

$$q \leftarrow \qquad r \leftarrow not\ s$$
$$p \leftarrow not\ q, s \qquad p \leftarrow not\ p$$

- Partial interpretation $\mathcal{J}$

True	Undefined	False
q	p, r	s

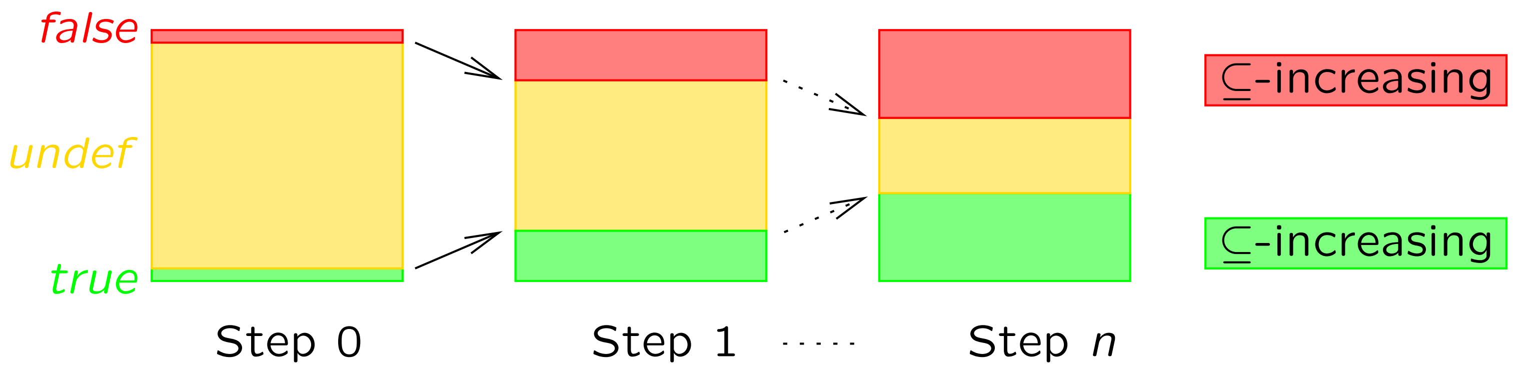
- Quotient program $\frac{P}{\mathcal{J}}$

$$q \leftarrow \qquad r \leftarrow true$$
$$p \leftarrow false, s \qquad p \leftarrow undef$$

- $\mathcal{I}$ is a partial model of $\frac{P}{\mathcal{J}}$ iff for all $Head \leftarrow Body$ in $\frac{P}{\mathcal{J}}$:
  - If $\mathcal{I}(Body) = true$ then $\mathcal{I}(Head) = true$
  - If $\mathcal{I}(Head) = false$ then $\mathcal{I}(Body) = false$
- **Least** partial model $LPM(\frac{P}{\mathcal{J}})$
  - $\mathcal{I}$ minimizes *true* atoms, and
  - $\mathcal{I}$ maximizes *false* atoms

True	Undefined	False
q, r	p	s

Well-Founded Models as Fixpoint Iteration



- Maintain two sets to represent $\mathcal{I}_i$:
  - The "*true*" atoms
  - The "*true* or *undef*" atoms
- Set $\mathcal{I}_0 =$ "all *undef*" and do $\mathcal{I}_{i+1} = LPM(\frac{P}{\mathcal{I}_i})$ until fixpoint, where
- seqeuence $(\mathcal{J}_0 =$ "all *false*"$), \mathcal{J}_1, \ldots, \mathcal{J}_{n-1}, (\mathcal{J}_n = \mathcal{J}_{n+1} = LPM(\frac{P}{\mathcal{I}_i}))$ obtained with operator associated to $(Head \leftarrow Body) \in \frac{P}{\mathcal{I}_i}$:

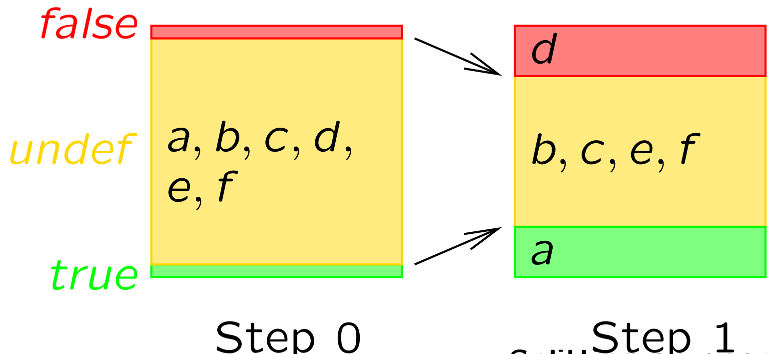
(i) If $\mathcal{J}_k(Body) = true$ then $\mathcal{J}_{k+1}(Head) = true$

(ii) If $\mathcal{J}_{k+1}(Head) = false$ then $\mathcal{J}_k(Body) = false$ iff

If $\underbrace{\mathcal{J}_k(Body) \neq false}_{\mathcal{J}_k(Body) \in \{true, undef\}}$ then $\underbrace{\mathcal{J}_{k+1}(Head) \neq false}_{\mathcal{J}_{k+1}(Head) \in \{true, undef\}}$

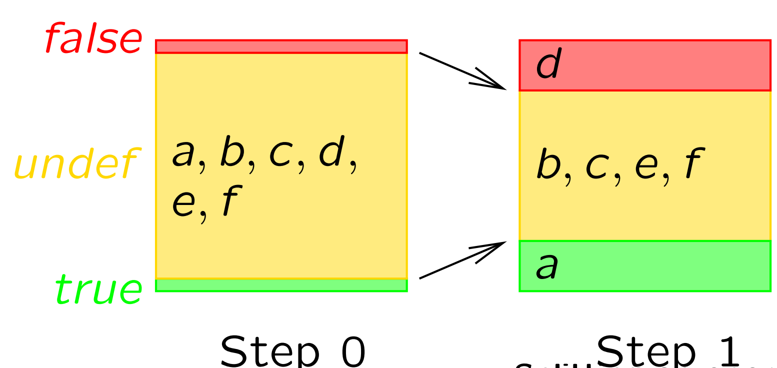
Computing Well-Founded Models, Step 0 $\mapsto$ Step 1

P
$a \leftarrow$
$c \leftarrow not\ b, a$
$b \leftarrow not\ c$
$e \leftarrow not\ d$
$f \leftarrow e$
$f \leftarrow not\ a$



Computing Well-Founded Models, Step 0 $\mapsto$ Step 1

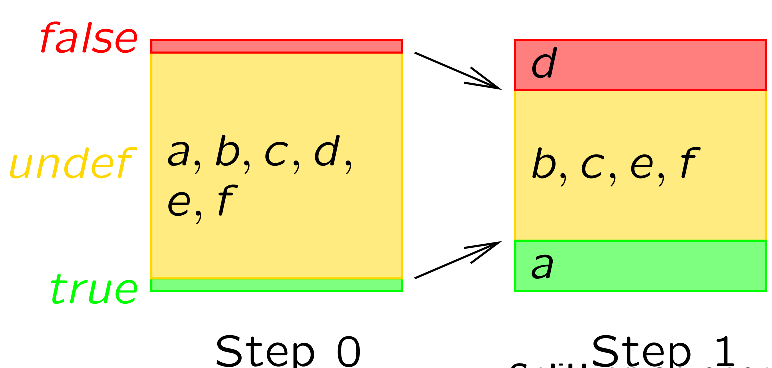
P	(i) build $P/$ [a, b, c, d, e, f]
$a \leftarrow$	$a \leftarrow$
$c \leftarrow \; not\ b, a$	$c \leftarrow undef, a$
$b \leftarrow \; not\ c$	$b \leftarrow undef$
$e \leftarrow \; not\ d$	$e \leftarrow undef$
$f \leftarrow \; e$	$f \leftarrow \; e$
$f \leftarrow \; not\ a$	$f \leftarrow undef$



Computing Well-Founded Models, Step 0 $\mapsto$ Step 1

P	(i) build $P/$ a, b, c, d, e, f
$a \leftarrow$	$a \leftarrow$
$c \leftarrow not\ b, a$	$c \leftarrow undef, a$
$b \leftarrow not\ c$	$b \leftarrow undef$
$e \leftarrow not\ d$	$e \leftarrow undef$
$f \leftarrow e$	$f \leftarrow e$
$f \leftarrow not\ a$	$f \leftarrow undef$

(ii) derive new *true* atoms a

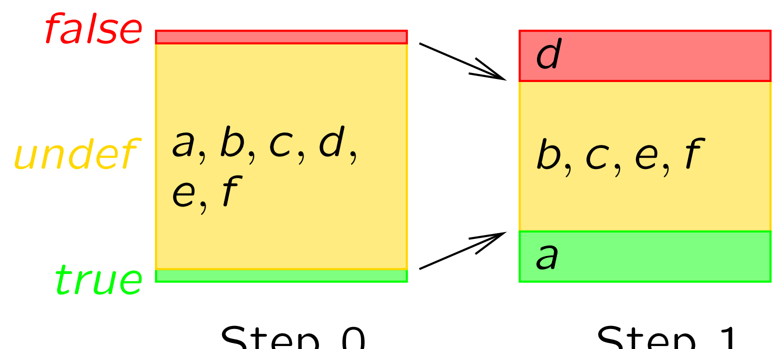


Computing Well-Founded Models, Step 0 $\mapsto$ Step 1

P	(i) build $P/$ [a, b, c, d, e, f]
$a \leftarrow$	$a \leftarrow$
$c \leftarrow not\ b, a$	$c \leftarrow undef, a$
$b \leftarrow not\ c$	$b \leftarrow undef$
$e \leftarrow not\ d$	$e \leftarrow undef$
$f \leftarrow e$	$f \leftarrow e$
$f \leftarrow not\ a$	$f \leftarrow undef$

(ii) derive new *true* atoms [a]

(iii) derive new *true* or *undef* atoms [a | b, c, e, f]



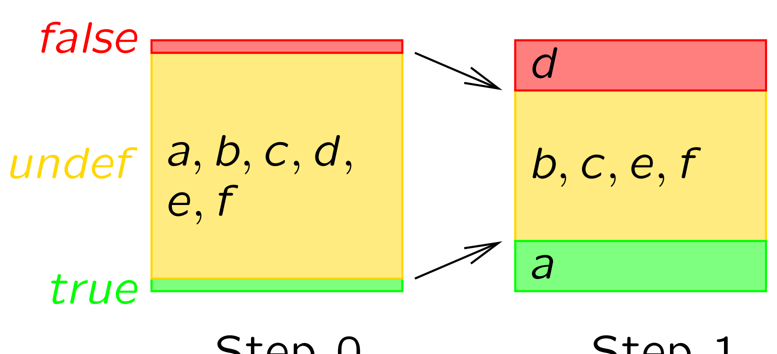
Computing Well-Founded Models, Step 0 $\mapsto$ Step 1

P	(i) build $P/$ [a, b, c, d, e, f]
$a \leftarrow$	$a \leftarrow$
$c \leftarrow not\ b, a$	$c \leftarrow undef, a$
$b \leftarrow not\ c$	$b \leftarrow undef$
$e \leftarrow not\ d$	$e \leftarrow undef$
$f \leftarrow e$	$f \leftarrow e$
$f \leftarrow not\ a$	$f \leftarrow undef$

(ii) derive new *true* atoms [a]

(iii) derive new *true* or *undef* atoms [a | b, c, e, f]

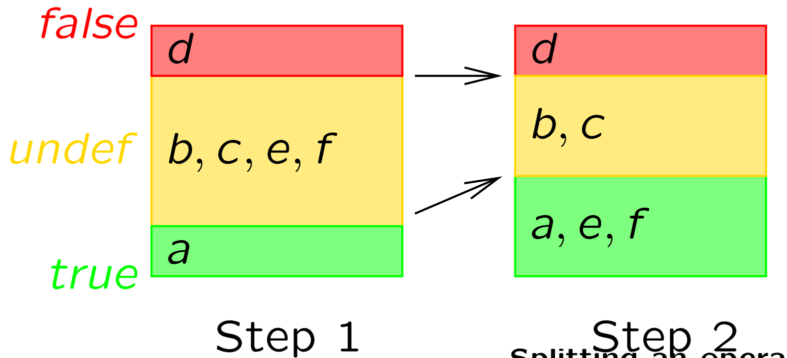
(iv) conclude new *false* atoms [d]



Computing Well-Founded Models, Step 1 $\longmapsto$ Step 2

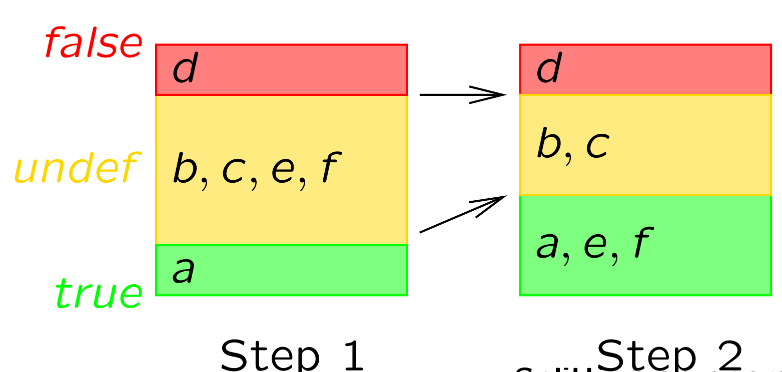
P

$$a \leftarrow$$
$$c \leftarrow not\ b, a$$
$$b \leftarrow not\ c$$
$$e \leftarrow not\ d$$
$$f \leftarrow e$$
$$f \leftarrow not\ a$$



Computing Well-Founded Models, Step 1 $\mapsto$ Step 2

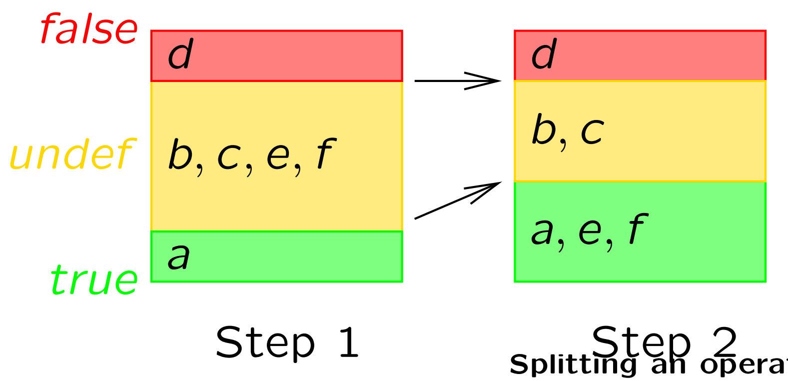
P	(i) build $P/$ a b, c, e, f d
$a \leftarrow$	$a \leftarrow$
$c \leftarrow \textit{not } b, a$	$c \leftarrow \textit{undef}, a$
$b \leftarrow \textit{not } c$	$b \leftarrow \textit{undef}$
$e \leftarrow \textit{not } d$	$e \leftarrow \textit{true}$
$f \leftarrow e$	$f \leftarrow e$
$f \leftarrow \textit{not } a$	$f \leftarrow \textit{false}$



Computing Well-Founded Models, Step 1 $\mapsto$ Step 2

P	(i) build $P/$ a | b, c, e, f | d
$a \leftarrow$	$a \leftarrow$
$c \leftarrow \textit{not } b, a$	$c \leftarrow \textit{undef}, a$
$b \leftarrow \textit{not } c$	$b \leftarrow \textit{undef}$
$e \leftarrow \textit{not } d$	$e \leftarrow \textit{true}$
$f \leftarrow e$	$f \leftarrow e$
$f \leftarrow \textit{not } a$	$f \leftarrow \textit{false}$

(ii) derive new *true* atoms a, e, f

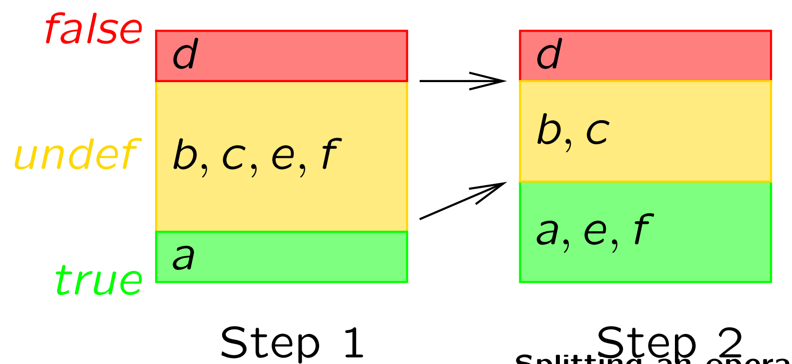


Computing Well-Founded Models, Step 1 $\longmapsto$ Step 2

P	(i) build $P/$ [a | b, c, e, f | d]
$a \leftarrow$	$a \leftarrow$
$c \leftarrow not\ b, a$	$c \leftarrow undef, a$
$b \leftarrow not\ c$	$b \leftarrow undef$
$e \leftarrow not\ d$	$e \leftarrow true$
$f \leftarrow e$	$f \leftarrow e$
$f \leftarrow not\ a$	$f \leftarrow false$

(ii) derive new *true* atoms [a, e, f]

(iii) derive new *true* or *undef* atoms [a, e, f \| b, c]



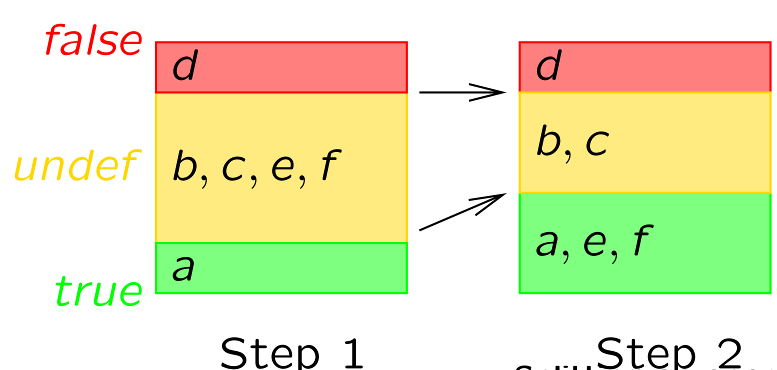
Computing Well-Founded Models, Step 1 $\mapsto$ Step 2

P	(i) build $P / \boxed{a}\boxed{b,c,e,f}\boxed{d}$
$a \leftarrow$	$a \leftarrow$
$c \leftarrow \; not\ b, a$	$c \leftarrow undef, a$
$b \leftarrow \; not\ c$	$b \leftarrow undef$
$e \leftarrow \; not\ d$	$e \leftarrow true$
$f \leftarrow \; e$	$f \leftarrow \; e$
$f \leftarrow \; not\ a$	$f \leftarrow false$

(ii) derive new *true* atoms $\boxed{a, e, f}$

(iii) derive new *true* or *undef* atoms $\boxed{a, e, f}\boxed{b, c}$

(iv) conclude new *false* atoms $\boxed{d}$



Abstraction Theory (Denecker, Marek and Truszczynsk

Recall Fitting operator for logic programs:

(i) If $\mathcal{I}_k(Body) = true$ then $\mathcal{I}_{k+1}(Head) = true$

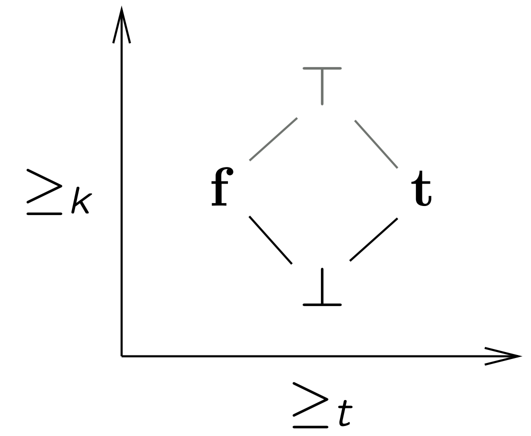
(ii) If If $\mathcal{I}_k(Body) \neq false$ then $\mathcal{I}_{k+1}(Head) \neq false$

Fitting: Semantics as fixpoints of certain derived operators

Abstraction Theory

- Operator (i) alone is sufficient, (ii) is derived (minor issue)
- Other major knowledge representation formalisms (Autoepistemic Logic, Default Logic) can be described by operators comparable to (i) with same monotonicity properties
- **Conclusion:** Develop theory on an abstract level.
- **Applications:**
  – Comparable (new) semantics for AEL and DL Logic as in logic programming
  – Abstract results on stratification

Ordering Interpretations



Ordering of truth values:

$\geq_k$ knowledge (precision, information) ordering
$\geq_t$ truth ordering

Maintain two sets $(X, Y) \in 2^\Sigma \times 2^\Sigma$ to represent an interpretation:

- The "*true*" atoms X
- The "*true* or *undef*" atoms Y

Further notions:

- (X, X) is **exact**
- (X, Y) is **consistent** iff $X \subseteq Y$

Ordering interpretations, bilattices $(2^\Sigma \times 2^\Sigma, \leq_k)$ and $(2^\Sigma \times 2^\Sigma, \leq_t)$:

$(X, Y) \leq_k (X', Y')$ iff $X \subseteq X'$ and $Y' \subseteq Y$ (Knowledge ordering)
$(X, Y) \leq_t (X', Y')$ iff $X \subseteq X'$ and $Y \subseteq Y'$ (Truth ordering)

Evaluation of Formulas

$$H_{(X,Y)}(\phi) = \begin{cases} \mathbf{t} & \phi \text{ is true in the interpretation defined by } (X, Y) \\ \mathbf{f} & \text{otherwise} \end{cases}$$

$$H_{(X,Y)}(p) = \begin{cases} \mathbf{t} & \text{if } p \in X \qquad (p \text{ an atom}) \\ \mathbf{f} & \text{otherwise} \end{cases}$$

$$H_{(X,Y)}(\phi \wedge/\vee \psi) = \begin{cases} \mathbf{t} & \text{if } H_{(X,Y)}(\phi) = \mathbf{t} \text{ and/or } H_{(X,Y)}(\phi) = \mathbf{t} \\ \mathbf{f} & \text{otherwise} \end{cases}$$

$$H_{(X,Y)}(\neg\phi) = \begin{cases} \mathbf{t} & \text{if } H_{(Y,X)}(\phi) = \mathbf{f} \\ \mathbf{f} & \text{otherwise} \end{cases}$$

Associating Operators to Programs

Let P be a Program. Define operator $U_P : 2^\Sigma \times 2^\Sigma \mapsto 2^\Sigma$:

$$U_P(X,Y) = \{p \in \Sigma \mid \text{there is } (p \leftarrow q, \neg r) \in P \text{ with } H_{X,Y}(q \wedge \neg r) = \mathbf{t}\}$$

Note: $H_{X,Y}(q \wedge \neg r) = \mathbf{t}$ iff q is *true* and r is *false* in (X, Y)

Special case

Well known two-valued operator $T_P : 2^\Sigma \mapsto 2^\Sigma$:

$$X \mapsto U_P(X,X)$$

Properties

- Fixpoints of T_P need not exist, take $P = \{p \leftarrow \neg p\}$
- Fixpoints of T_P are **two-valued supported models**
  E.g. fixpoints of $T_{\{p \leftarrow p\}}$ are $\{\}$ and $\{p\}$
- If P is definite then T_P is monotone; LFP is **minimal model**

Fitting Operator as Symmetric Application of U_P

Recall (X, Y) means (“*true* atoms”, “*true* or *undef* atoms”)

Recall

$$U_P(X, Y) = \{p \in \Sigma \mid \text{there is } (p \leftarrow q, \neg r) \in P \text{ with } H_{X,Y}(q \wedge \neg r) = \mathbf{t}\}$$

$H_{X,Y}(q \wedge \neg r) = \mathbf{t}$ iff q is *true* and r is *false* in (X, Y)

Now swap X and Y:

$$U_P(Y, X) = \{p \in \Sigma \mid \text{there is } (p \leftarrow q, \neg r) \in P \text{ with } H_{Y,X}(q \wedge \neg r) = \mathbf{t}\}$$

$H_{Y,X}(q \wedge \neg r) = \mathbf{t}$ iff q is *true* or *undef* and r is *false* or *undef* in (X, Y)

Define Fitting operator $\mathcal{T}_P(X, Y) = (U_P(X, Y), U_P(Y, X))$

$\mathcal{T}_P$ is $\leq_k$-monotone:

if $X \subseteq X'$ and $Y' \subseteq Y$

then $U_P(X, Y) \subseteq U_P(X', Y')$ and $U_P(Y', X') \subseteq U_P(Y, X)$

Intuition for $\mathcal{T}_P$

$$\mathcal{T}_P(X,Y)(p) = \begin{cases} \textit{true} & \text{if there is } (p \leftarrow q, \neg r) \in P \text{ where} \\ & q \text{ and } \neg r \text{ are } \textit{true} \text{ in } (X,Y) \\ \textit{true} \text{ or } \textit{undef} & \text{if there is } (p \leftarrow q, \neg r) \in P \text{ where} \\ & q \text{ and } \neg r \text{ are } \textit{true} \text{ or } \textit{undef} \text{ in } (X,Y) \\ \textit{false} & \text{otherwise} \end{cases}$$

Equivalently:

$$\mathcal{T}_P(X,Y)(p) = \begin{cases} \textit{true} & \text{if there is } (p \leftarrow q, \neg r) \in P \text{ where} \\ & q \text{ and } \neg r \text{ are } \textit{true} \text{ in } (X,Y) \\ \textit{false} & \text{if for all } (p \leftarrow q, \neg r) \in P \text{ it holds} \\ & q \text{ or } \neg r \text{ is } \textit{false} \text{ in } (X,Y) \\ \textit{true} \text{ or } \textit{undef} & \textit{otherwise} \end{cases}$$

Properties of $\mathcal{T}_P$

- $\mathcal{T}_P$ is $\leq_k$-monotone, thus least fixpoint exists;
  Bottom element is $(\{\}, \Sigma)$
  Gives **Kripke-Kleene semantics**, (or Fitting semantics)
- Examples

Program	Fixpoint iteration
$p \leftarrow \neg q$	$(\{\}, \{p, q\}) \rightarrow (\{\}, \{p\}) \rightarrow (\{p\}, \{p\})$
$p \leftarrow \neg p$	$(\{\}, \{p, q\}) \rightarrow (\{\}, \{p\})$
$p \leftarrow p$	$(\{\}, \{p, q\}) \rightarrow (\{\}, \{p\})$

Abstraction Theory (1)

- Given a lattice $(L, \leq)$ – concrete case $(2^\Sigma, \subseteq)$
- Bilattice $(L \times L, \leq_p)$ – concrete case $(2^\Sigma \times 2^\Sigma, \leq_k)$
- **Approximation:** any $\leq_p$-monotone operator $A : L \times L \mapsto L \times L$
  A can be written as

$$\underbrace{A(X,Y)}_{\mathcal{T}_P(X,Y)} = (\underbrace{A_1(X,Y)}_{U_P(X,Y)}, \underbrace{A_2(X,Y)}_{U_P(Y,X)})$$

- Derived operators (1) - holding an argument as parameter:

  $A^1(\cdot, Y) = \lambda X.A_1(X,Y)$ – concrete case $A^1(\cdot, Y) = \lambda X.U_p(X,Y)$
  $A^2(X, \cdot) = \lambda Y.A_2(X,Y)$ – concrete case $A^2(X, \cdot) = \lambda Y.U_p(Y,X)$

  Both A_1 and A_2 are $\leq$-monotone

Abstraction Theory (2)

- Derived operators (1) from above:

  $A^1(\cdot, Y) = \lambda X.A_1(X, Y)$
  $A^2(X, \cdot) = \lambda Y.A_2(X, Y)$

- Derived operators (2): $(C^{\downarrow}_{\mathcal{T}_P}(Y), C^{\uparrow}_{\mathcal{T}_P}(X)) = LPM(\frac{P}{(X,Y)})$

$$C^{\downarrow}_A(Y) = LFP(A^1(\cdot, Y))$$
$$C^{\uparrow}_A(X) = LFP(A^2(X, \cdot))$$

  Both $C^{\downarrow}_A$ and $C^{\uparrow}_A$ are $\leq$-antimonotone

- **Partial stable operator of A:**

$$\mathcal{C}_A(X, Y) = (C^{\downarrow}_A(Y), C^{\uparrow}_A(X))$$

  Because $C^{\downarrow}_A$ and $C^{\uparrow}_A$ are $\leq$-antimonotone, $\mathcal{C}_A$ is $\leq_p$-monotone

  $LFP(\mathcal{C}_{\mathcal{T}_P})$ (wrt. $\leq_k$) is the **well-founded model**
  Two-valued fixpoints of $\mathcal{C}_{\mathcal{T}_P}$ are the **stable models**

Summary - Abstraction Theory $\rightarrow$ Logic Programming

Start with an operator O – concrete case U_P.

Semantics of derived operators:

- $T_P(X) = U_P(X, X)$
  - Fixpoints: 2-valued supported models
- $\mathcal{T}_P(X, Y) = (U_P(X, Y), U_P(Y, X))$
  - Fixpoints: 3-valued supported models
  - LFP: Kripke-Kleene semantics
- Let $A = \mathcal{T}_P$. Partial stable operator $\mathcal{C}_A(X, Y) = (C_A^{\downarrow}(Y), C_A^{\uparrow}(X))$
  - Fixpoints: (partial) stable models
  - LFP: well-founded model

Application to Default Logic and Autoepistemic Logic

Default Logic and Autoepistemic Logic semantics can be described by suitable operators O. Then:

- Usual Moore semantics for AEL is given by 2-valued supported models ("$X \mapsto U_P(X,X)$")
- Usual Reiter semantics for DL is given by 2-valued stable models
- Intuitive mapping from DL to AEL:

Default logic inference rule:	Translation to Autoepistemic Logic:
$\dfrac{\alpha : \beta_1, \ldots, \beta_n}{\gamma}$	$\mathbf{L}\alpha \land \neg\mathbf{L}\neg\beta_1 \land \cdots \land \neg\mathbf{L}\neg\beta_n \rightarrow \gamma$

Reiter semantics for DL is the same as the 2-valued stable model semantics for the translation!

Dependency Graph leads to Stratification

Example, $\Sigma = \{p, q, r\}$:

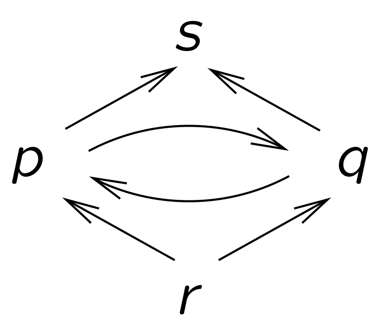
P:

$$s \leftarrow p, q \qquad (1)$$

$$p \leftarrow \neg q, \neg r \qquad (2)$$

$$q \leftarrow \neg p, \neg r \qquad (3)$$

Dependency graph:



$$\Sigma_2 = \{s\}$$

$$\Sigma_1 = \{p, q\}$$

$$\Sigma_0 = \{r\}$$

Suggests **splitting** $\Sigma = \Sigma_0 \dot{\cup} \Sigma_1 \dot{\cup} \Sigma_2$

Contribution: The program P is not stratified in the standard sense, but models can still be constructed in a stratified way $\Sigma_0 \rightarrow \Sigma_1 \rightarrow \Sigma_2$.

Stratification in Abstraction Theory - Product Lattices

So far: lattice $(2^\Sigma, \subseteq)$ and bilattice $(2^\Sigma \times 2^\Sigma, \leq_k)$

Now:

- Product lattice $(\bigotimes_{i=0,\ldots,n} 2^{\Sigma_i}, \subseteq)$, where
  - $(\bigotimes_{i=0,\ldots,n} 2^{\Sigma_i}, \subseteq) = (2^{\Sigma_0}, \ldots, 2^{\Sigma_n})$, and
  - $(x_0, \ldots, x_n) = x \subseteq y = (y_0, \ldots, y_n)$ iff $x_0 \subseteq y_0$ and ... and $x_n \subseteq y_n$
- Example: $\Sigma = \underbrace{\{r\}}_{\Sigma_0} \dot\cup \underbrace{\{p, q\}}_{\Sigma_1} \dot\cup \underbrace{\{s\}}_{\Sigma_2}$

  $x = (\{r\}, \{p\}, \{\}) \in \bigotimes_{i=0,1,2} 2^{\Sigma_i}$
  $y = (\{r\}, \{p, q\}, \{s\}) \in \bigotimes_{i=0,1,2} 2^{\Sigma_i}$
  It holds $x \subseteq y$
- Bilattice of product lattices $(\bigotimes_{i=0,\ldots,n} 2^{\Sigma_i} \times \bigotimes_{i=0,\ldots,n} 2^{\Sigma_i}, \text{“}\leq_k\text{”})$
- Product lattice of bilattices $(\bigotimes_{i=0,\ldots,n}(2^{\Sigma_i} \times 2^{\Sigma_i}), \text{“}\leq_k\text{”})$

Stratification in Abstraction Theory - Results

Notation: e.g. $x = (\{r\}, \{p\}, \{\})$. Then $x|_{\leq 1} = (\{r\}, \{p\})$

Definition: ("Applying O at stratum i does not depend from strata $> i$.")

Operator O on a product lattice L is **stratifiable** iff
for all $x, y \in L$ and all $i = 0, \ldots, n$:
if $x|_{\leq i} = y|_{\leq i}$ then $O(x)|_{\leq i} = O(y)|_{\leq i}$.

Theorem: ("Logic programming: splitting results in stratification")

Let P be a logic program and $(\Sigma_i)_{i=0,\ldots,n}$ a splitting.
Then the operator $\mathcal{T}_P$ on the bilattice of the product lattice
$(\bigotimes_{i=0,\ldots,n} 2^{\Sigma_i} \times \bigotimes_{i=0,\ldots,n} 2^{\Sigma_i}, \text{“}\leq_k\text{”})$ is stratifiable.

Theorem: ("Stratum-wise computation of fixpoints")

Let L be a product lattice, O a stratifiable operator and $x \in L$.
Then x is a fixpoint of O iff for all $i = 0, \ldots, n$:
$x|_i$ is a fixpoint of $O(x)|_i$ $\quad$ ($x|_i$ fixpoint of $O_i^{x|_{<i}}$).

$\longrightarrow$ similar result for least fixpoints

Stratification: Example

O is $\mathcal{T}_P$, where

P:

$$s \leftarrow p, q \quad (1)$$

$$p \leftarrow \neg q, \neg r \quad (2)$$

$$q \leftarrow \neg p, \neg r \quad (3)$$

Task: compute well-founded model x of P (i.e. least fixpoint of $\mathcal{T}_P$)

Construct well-founded models of $P_0^{x|_{<0}}$, $P_1^{x|_{<1}}$, $P_2^{x|_{<2}}$

$\Sigma_0 = \{r\}$, $P_0 = \emptyset$, $P_0^{x|_{<0}} = \emptyset$, well-founded model is $x|_{<1} = (\{\}, \{\})$

$\Sigma_1 = \{p, q\}$, $P_1 = \{(2), (3)\}$, with $x|_{<1}(r) =$ *false* have

$P_1^{x|_{<1}}$:

$$p \leftarrow \neg q, \mathbf{t} \quad (2')$$

$$q \leftarrow \neg p, \mathbf{t} \quad (3')$$

Well-founded model is $x|_{<2} = ((\{\}, \{\}), (\{\}, \{p, q\}))$

Stratification: Example

O is $\mathcal{T}_P$, where

P:

$$s \leftarrow p, q \tag{1}$$

$$p \leftarrow \neg q, \neg r \tag{2}$$

$$q \leftarrow \neg p, \neg r \tag{3}$$

Recall well-founded model $x|_{<2} = ((\{\}, \{\}), (\{\}, \{p, q\}))$

$\Sigma_2 = \{s\}$, $P_2 = \{(1)\}$,
with $x|_{<2}(r) = \mathit{false}$, $x|_{<2}(p) = \mathit{undef}$ and $x|_{<2}(q) = \mathit{undef}$ have

$P_2^{x|_{<2}}$:

$$s \leftarrow \mathbf{u}, \mathbf{u} \tag{1'}$$

Well-founded model is $x|_{<3} = ((\{\}, \{\}, \{\}), (\{\}, \{p, q\}, \{s\}))$

This is the well-founded model of P

Conclusions

- Abstraction theory: framework to explain and construct semantics of knowledge representation formalism in a uniform way
- Abstract concept of stratification: useful for own work