Model Elimination Without Contrapositives

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Abstract. We present modifications of model elimination which do not necessitate the use of contrapositives. These restart model elimination calculi are proven sound and complete. The corresponding proof procedures are evaluated by a number of runtime experiments and they are compared to other well known provers. Finally we relate our results to other calculi, namely the connection method, modified problem reduction format and Near-Horn Prolog.

1 Introduction

This paper demonstrates that model elimination can be defined such that it is complete without the use of contrapositives. We believe that this result is interesting in at least two respects: it makes model elimination available as a calculus for non-Horn logic programming and it enables model elimination to perform proofs of mathematical theorems by case analysis.

Let us first explain what we mean by the term "without the use of contrapositives". In implementations of theorem proving systems usually n procedural counterparts $L_i \leftarrow \overline{L}_1 \wedge \cdots \wedge \overline{L}_{i-1} \wedge \overline{L}_{i+1} \wedge \cdots \wedge \overline{L}_n$ for a clause $L_1 \vee \cdots \vee L_n$ have to be considered. Each of these is referred to as a *contrapositive* of the given clause and represents a different entry point during the proof search into the clause. It is well-known that for Prolog's SLD-resolution one single contrapositive suffices, namely the "natural" one, selecting the head A of the clause $A \vee \neg B_1 \vee \cdots \vee \neg B_n$ as entry point. For full first-order systems the usually required n contrapositives are either given more *explicitly* (as in the SETHEO prover [LSBB92]) or more *implicitly* (as in the connection method by allowing to set up a connection with *every* literal in a clause [Ede92]). The distinction is merely a matter of presentation and will be given up for this paper. Now, by a system "without contrapositives" we mean more precisely a system which does not need all n contrapositives for a given n-literal clause.

Model elimination [Lov68] is a calculus, which is the base of numerous proof procedures for first order deduction. There are high speed theorem provers, like METEOR [AS92] or SETHEO [LSBB92]. The implementation of model elimination provers can take advantage of techniques developed for Prolog. For instance, Stickel's Prolog technology theorem proving system (PTTP, [Sti88]) uses Horn clauses as an intermediate language. Hence, it should be straightforward to use model elimination and PTTP in the context of **non-Horn logic programming**. Indeed this possibility is discussed in

various papers; however it is discarded by some authors because of the necessity to use contrapositives (e.g. [Lov91, Pla88]). The argument is given by Plaisted [Pla88] explicitly: "In general, however, we feel that the need for contrapositives makes it difficult to view model elimination as a true programming language in the style of Prolog, since the user has less control over the search." Suppose, for example we are given an input clause[1]

$$prove(and(X, Y)) \leftarrow prove(X) \wedge prove(Y)$$

which can be used within a formalization of propositional calculus. A possible contrapositive is

$$\neg prove(X) \leftarrow \neg prove(and(X, Y)) \wedge prove(Y)$$

The procedural reading of this contrapositive is somewhat strange and leads to an unnecessary blowing-up of the search space; in order to prove $\neg prove(X)$ one has to prove $prove(Y)$ – a goal which is totally unrelated to $\neg prove(X)$ by introducing a new variable. Such observations had been the motivation for the development of calculi which need no contrapositives, e.g. Loveland's NearHorn-Prolog. Gabbay's N-Prolog [Gab85] when restricted to clause logic is general enough to be instantiated to both NearHorn-Prolog and problem reduction formats ([Pla88], see also Section 5 below; [RL92] contains a comparison of these).

Another motivation for the new calculi is as follows: in proving theorems such as "if $x \neq 0$ then $x^2 > 0$" a human typically uses case analysis according to the axiom $X < 0 \vee X = 0 \vee -X < 0$. This seems a very natural way of proving the theorem and leads to a well-understandable proof. Our modified model elimination procedure carries out precisely such a proof by case analysis. Experimental results with similar examples from calculus and a comparison with other proof procedures are presented in this paper.

The calculi we derive in this paper are a very small modification of model elimination and hence allow for Prolog implementation techniques. They are complete without the use of contrapositives and hence well-suited for logic programming and for theorem proving by "case analysis" or "splitting".

As a more theoretical contribution we will show that one of the NearHorn-Prologs, namely InH-Prolog ([LR89]), can be seen as one of our modified model elimination procedures.

As a final point, we discovered that the connection method ([Bib87, Ede92], [BF93] contains a comparative study) is complete without contrapositives and *without any change to the calculus*. This surprising result is due to a relaxed complementary-literal condition which subsumes the above-mentioned small change in model elimination.

This paper is organised as follows: in the following section we review the model elimination calculus we use as a starting point of our investigation. In section 3 we define various variants of this calculus and give soundness and completeness proofs of the weakest one. In section 4 we give some experimental results with a PTTP-implementation and in section 5 we discuss related work.

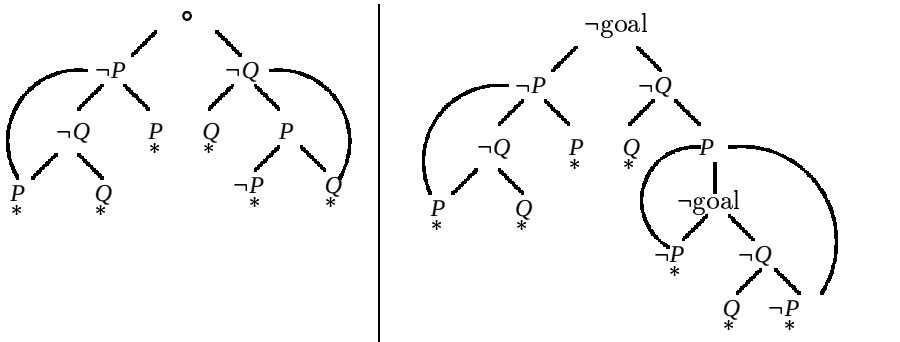
[1] Taken from [Pla88].

2 Review of Tableau Model Elimination

As a starting point we use a model elimination calculus that differs from the original one presented by [Lov68]; it is described in [LSBB92] as the base for the prover SETHEO. In [BF93] this calculus is discussed in detail by presenting it in a consolution style [Ede91] and comparing it to various other calculi. This model elimination manipulates trees by extension- and reduction-steps. In order to recall the calculus and to state a running example consider the clause set

$$\{\{P, Q\}, \{\neg P, Q\}, \{\neg Q, P\}, \{\neg P, \neg Q\}\},$$

A model-elimination refutation is depicted in Figure 1 (left side). It is obtained by successive fanning with clauses from the input set (*extension steps*). Additionally, it is required that every inner node (except the root) is complementary to one of its sons. An arc indicates a *reduction step*, i.e. the closing of a branch due to a path literal complementary to the leaf literal.



A Model Elimination Refutation.

A Restart Model Elimination Refutation (positive *goal*-nodes are not displayed).

Fig. 1. Model Elimination (left side) vs. Restart Model Elimination (right side) as defined in Section 3.

In the following we use a formal presentation of the calculus along the lines of [BF93]. Instead of trees we manipulate multisets of paths, where paths are sequences of literals.

A clause is a multiset of literals, usually written as the disjunction $L_1 \vee \ldots \vee L_n$. A *connection* in a set of clauses is a pair of literals, written as (K, L), which can be made complementary by application of a substitution. A *path* is a sequence of literals, written as $p = \langle L_1, \ldots, L_n \rangle$. L_n is called the *leaf* of p, which is also denoted by $\mathit{leaf}(p)$; similarly, the first element L_1 is also denoted by $\mathit{first}(p)$. `∘' denotes the append function for literal sequences. Multisets of paths are written with caligraphic capital letters.

Definition 1. (Tableau Model Elimination) Given a set of clauses S.

– The inference rule *extension* is defined as follows:

$$\frac{\mathcal{P} \cup \{p\} \qquad\qquad L \vee R}{\mathcal{R}} \quad \text{, where}$$

1. $\mathcal{P} \cup \{p\}$ is a path multiset, and $L \vee R$ is a variable disjoint variant of a clause in S; L is a literal and R denotes the rest literals of $L \vee R$.
2. $(\mathit{leaf}(p), L)$ is a connection with MGU σ.
3. $\mathcal{R} = (\mathcal{P} \cup \{p \circ \langle K \rangle \mid K \in R\})\sigma$.

– The inference rule *reduction* is defined as follows:

$$\frac{\mathcal{P} \cup \{p\}}{\mathcal{P}\sigma} \quad \text{, where}$$

1. $\mathcal{P} \cup \{p\}$ is a path multiset, and
2. there is a literal L in p such that $(L, \mathit{leaf}(p))$ is a connection with MGU σ.

– A sequence $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ is called a *model elimination derivation* iff
$\mathcal{P}_1$ is a path multiset $\{\langle L_1 \rangle, \ldots, \langle L_n \rangle\}$ consisting of paths of length 1, with $L_1 \vee \ldots \vee L_n$ in S (also called the *goal clause*), and
$\mathcal{P}_{i+1}$ is obtained from $\mathcal{P}_i$ by means of an extension step with an appropriate clause C, or
$\mathcal{P}_{i+1}$ is obtained from $\mathcal{P}_i$ by means of a reduction step.

The path p is called *selected path* in both inference rules. Finally, a *refutation* is a derivation where $\mathcal{P}_n = \{\}$.

Note that this calculus does not assume a special selection function which determines which path is to be extended or reduced next. Correctness and completeness of this calculus follows immediately from a result proved in [Bau92].

3 Restart Model Elimination Calculi

Let us now modify the calculus given above, such that no contrapositives are needed.

In order to get a complete calculus, we have to assume that there exists only one goal, i.e. a clause containing only negative literals, which furthermore does not contain variables. Without loss of generality this can be achieved by introducing a new clause $\leftarrow \mathit{goal}$ where goal is a new predicate symbol and by modifying every purely negative clause $\neg B_1 \vee \cdots \vee \neg B_n$ to $\mathit{goal} \leftarrow B_1, \ldots, B_n$. In the following we will refer to clause sets S satisfying that property as clause sets in *goal-normal form*. Note that since the goal literal is attached only to purely negative clauses the Horn status of the given clause set is not affected by the transformation. Furthermore, besides the goal-normal form we will only allow derivations which start with the goal clause $\leftarrow \mathit{goal}$.

Soundness of this transformation is evident. Completeness holds as follows:

Theorem 2. (Completeness of Model Elimination) *Let S be an unsatisfiable clause set in goal-normal form. Then there exists a tableau model elimination refutation of S with goal clause* $\leftarrow$ *goal. Furthermore, if S is Horn then no reduction steps are required in this refutation.*

We are now ready to modify the calculus, such that no contrapositives are necessary. This will be done in three steps: as a base we define a *restart model elimination* by a small modification in the definition of tableau model elimination, with the result that no contrapositives are needed. Then we weaken this calculus by introducing a *selection function*, which determines which positive head literal can be used for an extension step. Finally, as a further weakening we introduce *strict restart model elimination* by disallowing reduction steps with a positive leaf literal.

A completeness proof is given for the weakest variant, i.e. strict restart model elimination with selection function. Completeness of the stronger variants follows from this result as a simple corollary.

Definition 3. (Restart Model Elimination) Assume the following line additionally given between conditions 1 and 2 in the definition of *extension* (Def. 1).

1a. if $leaf(p)$ is positive then let $p := p \circ \langle first(p) \rangle$ in conditions 2 and 3.

The calculus of *restart model elimination* consists of thus modified extension inference rule and of the reduction inference rule. An extension step with lengthening according to 1a above is also called a *restart*.

If clauses are written in a sequent style $A_1, \ldots, A_n \leftarrow B_1, \ldots, B_m$ then it is clear that, for syntactical reasons extension steps are possible only with head literals A's and not with B's from the body. Thus it is possible to represent clauses as above *without* the need of augmenting them with all contrapositives; only contrapositives with conclusions (i.e. entry points) stemming from the positive literals are necessary.

The price of the absence of contrapositives is that whenever a path ends with a positive literal, the root of the tree, i.e. the clause $\neg goal$ has to be copied and the path has to be lengthened with that literal. Note that there is only a restriction on the applicability of extension steps – reductions are still allowed when a path ends with a positive literal.

In Figure 1 (left side) there is one extension step, which is no longer allowed in restart model elimination, namely the extension of the path $\langle \neg Q, P \rangle$. Note that with our assumptions on goals, this path becomes $p = \langle \neg goal, \neg Q, P \rangle$ in restart model elimination. There is no reduction step possible and since $leaf(p)$ is positive, we lengthen p to $p' = \langle \neg goal, \neg Q, P, \neg goal \rangle$ in a restart step, which can finally be extended to the path multiset

$$\{\langle \neg goal, \neg Q, P, \neg goal, \neg P \rangle, \langle \neg goal, \neg Q, P, \neg goal, \neg Q \rangle\}$$

The complete restart model elimination refutation is depicted in Figure 1 (right side).

It is obvious that for reasons of efficiency a proof procedure based on this calculus must provide some refinements. For example, the use of lemmas or factoring might reduce the amount of redundancy introduced by restart steps. In the example from

Figure 1 (right side) the restart led to the newly introduced paths ending with $\neg P$ and $\neg Q$, respectively. When processing the tree from left to right it is obvious that solving $\neg P$ would be unnecessary, since there is already a closed subtree containing $\neg P$ as a root; thus a proof procedure would benefit extremely from the possibility of using lemmas or factoring [LMG93]. However, we are lucky, the same effect could be achieved by a reduction step. These and other topics concerning proof procedures are discussed in the following section.

Selection Function

Now we weaken the calculus by introducing a selection function on head literals.

Definition 4. (Selection Function) A *selection function* f is a function that maps a clause $A_1, \ldots, A_n \leftarrow B_1, \ldots, B_m$ with $n \geq 1$ to an atom $L \in \{A_1, \ldots, A_n\}$. L is called the *selected literal* of that clause by f. The selection function f is required to be *stable under lifting* which means that if f selects $L\gamma$ in the instance of the clause $(A_1, \ldots, A_n \leftarrow B_1, \ldots, B_m)\gamma$ (for some substitution γ) then f selects L in $A_1, \ldots, A_n \leftarrow B_1, \ldots, B_m$.
Now let f be a selection function, and assume the following line additionally given between conditions 2 and 3 in the definition of *restart* (Def. 3).

2a L is selected in C by f.

This modified calculus is called *(restart) model elimination with selection function*.

Assume there is a path p in our running example with $leaf(p) = \neg P$ and the selection function gives $f(P, Q \leftarrow) = Q$, then it is not allowed to perform an extension step with $P, Q \leftarrow$. Positive literals not selected by f can only be used within reduction steps. Note that the proof in Figure 1 (right side) is a proof with the above assumed selection function.

Strict Restart Model Elimination

As a further restriction we force the calculus to perform restarts whenever they are possible, i.e. if a leaf of a path is a positive literal it may not be used for a reduction step. Since for these leaves extension steps are possible only through a restart, we call this calculus strict restart model elimination. This restriction is motivated in several ways: first, a comparable restriction is formulated within Plaisted's modified problem reduction format ([Pla88], see also Section 5 below) and we would like to evaluate it within our framework; second, strict restart model elimination minimizes the search in ancestor lists, which occasionally results in shorter runtimes to find a proof. See also [Pla90] for restrictions on accessing ancestor lists within a non-restart calculus.

Definition 5. (Strict Restart Model Elimination) The inference rule *reduction* is modified by adding the following line after condition 2 to the definition of *reduction* (Def. 1).

3. and $leaf(p)$ is a negative literal.

Such reduction steps are called *strict reduction steps*. *Strict restart model elimination* is defined to be the same as restart model elimination, except that "reduction step" is replaced by "strict reduction step".

For the rest of this section we deal with soundness and completeness of restart model elimination. Soundness of the calculus follows immediately by showing that every restart model elimination proof can be mapped to a proof in the free variable semantic tableau calculus ([Fit90]), while completeness will be proven directly.

Theorem 6. (Soundness) *Restart model elimination is sound.*

Theorem 7. (Completeness) *Let f be a selection function and S be a clause set in goal-normal form. Then there exists a strict restart model elimination refutation of S with goal $\leftarrow$ goal and selection function f.*

Since a strict reduction step is by definition also a reduction step we obtain as a corollary the completeness of the non-strict restart model elimination.

Since a selection function restricts the set of permissible derivations, completeness without selection function follows immediately from completeness with selection function.

Here we restrict to the proof on the ground level (Lemma 8 below). Although not quite trivial, lifting can be carried out by using standard techniques. In particular, by stability under lifting (Def. 4) it is guaranteed that the selection function will select on the first order level a literal whose ground instance was selected at the ground level.

Ground completeness reads as follows:

Lemma 8. (Ground Completeness) *Let f be a selection function and S be an unsatisfiable ground clause set in goal-normal form. Then there exists a strict restart model elimination refutation with selection function f of S with goal clause $\leftarrow$ goal.*

Proof. Informally, the proof is by splitting the non-Horn clause set into Horn sets, assuming by completeness of model elimination refutations without reduction steps, and then assembling these refutations into the desired restart model elimination refutation. There, reduction steps come in by replacing extension steps with split unit clauses by reduction steps to the literals where the restart occurred.

For the formal proof some terminology is introduced: we say that a path multiset $\mathcal{P}$ "contains (an occurrence of) a clause $A_1, \ldots, A_n \leftarrow B_1, \ldots, B_m$" iff for some path p it holds $\{p \circ \langle A_1 \rangle, \ldots, p \circ \langle A_n \rangle, p \circ \langle \neg B_1 \rangle, \ldots, p \circ \langle \neg B_m \rangle\} \subseteq \mathcal{P}$ If we speak of "replacing a clause C in a derivation by a clause $C \cup D$" we mean the derivation that results when using the clause $C \cup D$ in place of C in extension steps. Also, the same literal $L \in C$ must be used to constitute the connection.

By a "derivation of a clause C" we mean a derivation that ends in a path multiset which contains (several occurrences of) the clause C.

Let $k(S)$ denote the number of occurrences of positive literals in S minus the number of definite clauses[2] in S ($k(S)$ is related to the *excess literal parameter* in [AB70]). Now we prove the claim by induction on $k(S)$.

[2] A *definite clause* is a clause containing exactly one positive literal.

Induction start ($k(S) = 0$): M must be a set of Horn clauses. By Theorem 2 there exists a model elimination refutation of S with goal $\leftarrow goal$ without reduction steps. Furthermore, for syntactical reasons, in every extension step only the single positive literal (and never a negative literal) of the extending clause can be selected. Thus, this refutation is also a strict restart model elimination refutation.

Induction step ($k(S) > 0$): As the induction hypothesis suppose the result to hold for unsatisfiable ground clause sets S' in goal-normal form with $k(S') < k(S)$.

Since $k(S) > 0$, S must contain a non-Horn clause $C = A_1, A_2, \ldots, A_n \leftarrow B_1, \ldots, B_m$ with $n \geq 2$. W.l.o.g. assume that A_1 is the literal selected by f in C. Now define n sets

$$\begin{aligned} S_1 &:= (S \setminus C) \cup \{A_1 \leftarrow B_1, \ldots, B_m\} \\ S_2 &:= (S \setminus C) \cup \{A_2\} \\ &\vdots \\ S_n &:= (S \setminus C) \cup \{A_n\} \end{aligned}$$

Every set S_i ($i = 1 \ldots n$) is unsatisfiable (because otherwise, a model for one of them would be a model for S). Furthermore, it holds $k(S_i) = k(S) - n + 1 < k(S)$. Thus, by the induction hypothesis there exist strict restart model elimination refutations R_i with goal clauses $\leftarrow goal$ of S_i, respectively.

Now consider R_1 and replace in R_1 every occurrence of the clause $A_1 \leftarrow B_1, \ldots, B_m$ by C. Call this derivation R_1'. Since A_1 is the sole positive literal in $A_1 \leftarrow B_1, \ldots, B_m$, A_1 must have been selected in the extension steps with that clause in R_1. Thus the corresponding extension steps in R_1' with C are legal in the sense of the restriction to the selection function.

R_1' is a derivation of, say, k_j occurrences of the positive unit clauses A_j ($j = 2 \ldots n$) from the input set S. Now every A_j can be eliminated from the input set S according to the following procedure: for $j = 2 \ldots n$ append R_{j-1}' k_j times with the refutation R_j, however with the first extension step in R_j being replaced by a restart step at one of the paths ending A_j. Note here that the restart step produces exactly the same goal clause $\leftarrow goal$ as is in R_j. Let R_j'' be the refutation resulting from these k_j restart steps at leaf A_j. R_j'' is a restart model elimination refutation of $S \cup \{A_j, \ldots, A_n\}$. In order to turn R_j'' into a refutation of $S \cup \{A_{j+1}, \ldots, A_n\}$, replace every extension step with A_j in the appended refutations R_j by a reduction step to the positive path literal A_j. The resulting refutation R_j' is a desired strict model elimination refutation of $S \cup \{A_{j+1}, \ldots, A_n\}$.

Finally, R_n' is the desired strict restart model elimination refutation of S alone.

Regularity in Restart Theory Model Elimination.

Regularity means for ordinary model elimination that it is never necessary to construct a tableau where a literal occurs more than once along a path. Regularity tends to be one of the more useful refinements of model elimination. Unfortunately, regularity is *not* compatible to restart model elimination. This can be seen easily as the $goal$-literal is copied in restart steps, thus violating the regularity restriction. Hence at least the goal literal has to be excluded from the regularity restriction, because otherwise restart steps

are impossible! But even with this exception completeness is lost in general, since after a restart step it might be necessary to repeat – in parts – a proof found so far up to the restart step.

However the following observations allows to define a somewhat weaker notion of regularity: First, the proof of Lemma 8 proceeds by splitting the input clause set into Horn sets and then assembles the existing non-restart refutations $R_1, \ldots R_n$ into a restart refutation R'_n. Since this assembling is done "blockwise" (the R_is are not interleaved among each other and keep their structure) some properties of the R_is carry over to their respective occurrence in R'_n. In particular, the *regularity* of the R_is carries over in this way. Hence we define a path as *blockwise regular (version 1)* iff every pair of occurrences of identical literals (unequal to $\neg\, goal$) is separated by at least one occurrence of the literal $\neg\, goal$. A derivation is is called *blockwise regular* iff every path in every of its path multisets is blockwise regular. From these considerations and definitions it follows with Lemma 8 that this restriction is complete. Since all the R_is are refutations of *Horn* clause sets this regularity restriction applies only to *negative* literals along a path. Thus we might derive a blockwise regular path $p = \neg goal \cdots A \cdots \neg goal \cdots A$. We wish to extend blockwise regularity to forbid such duplicate occurrences of positive literals, and say that a branch is *positive regular* iff all the positive literals occurring in it are pairwise distinct (not identical). Extending the preceding definition, we define a branch to be *blockwise regular (final version)* iff it is blockwise regular (version 1) and positive regular. Fortunately it holds:

Theorem 9. *Restart model elimination is complete when restricted to blockwise regular refutations (final version).*

4 PTTP without Contrapositives – Experimental results

The reader familiar with Stickel's PTTP-technique ([Sti88]) may have already noticed that the restart variant can be implemented very easily using the PTTP-technique. Indeed, we implemented the restart model elimination calculus in the theorem proving system PROTEIN ([BF94]). Both the implementation language and the target language for the compiled code is ECL^{i}PS^e, an enhanced Prolog-dialect.

We ran several examples known from the literature, and some new ones. We compared several versions of the prover, varying in strict restart model elimination vs. (non-strict) restart model elimination, and selection function vs. no selection function; the first four columns in Figure 3 contain the runtime results. Column 5 (**MPRF**) contains the results for the *Modified Problem Reduction Format* prover (see also the section on related work below). The option "nosave flag cleared" means that caching is enabled. The data, taken from [Pla88], were obtained on a SUN 3 workstation, whereas the other provers ran on a SUN Sparc 10/40. Hence, for normalisation the times for the MPRF prover were divided by 14.

The default flag settings in our provers include *ground reduction steps* (in reduction steps where no substitution is involved no further proof alternatives need to be explored) and *blockwise regularity* as defined at the end of Section 3. Other flags allow for the generation of *(Unit)-lemmas* (currently *all* lemma candidates are stored) and *factoring*. Unless otherwise noted, in Figure 3 the default flags are used.

For iterative deepening, the threshold was increased in each iteration by 1, and each extension step was uniformly charged with a cost of 1.

Furthermore, we also found it interesting to run standard model elimination provers which use contrapositives ("PTTP" and "Setheo", the right two columns in Figure 3). This demonstrate that in some cases, namely the examples from real-analysis, restart model elimination results in better performance, whereas in the usual benchmarks the results with restart based procedures are often in the same order of magnitude.

Now let us summarize the results. Compared among each other, each of the 4 versions of restart model elimination has its justification by dedicated examples. In the parameter space *non-strict restart* vs. *strict restart* model elimination we prefer as the default strategy the *non-strict* version, since whenever the *strict* version found a proof in reasonable time, the *non-strict* version did as well; but on most examples the *strict* version failed or behaved poorly, while the other version found a proof.

	Restart Model Elimination		**Strict Restart Model Elimination**		**MPRF**	**ME**	**ME**
Example	w/o Selection	w. Selection	w/o Selection	w. Selection		PTTP	Setheo
Non-Obvious `MSC/MSC006-1`	**2.7**	∞	∞	∞	128 3.2 [4]	0.3	0.5
Eder45	3.4 1.7 [1]	1.8/10.1 [3] 2.5 [1]	0.5 0.7 [1]	3.1/8.5 [3] 0.9 [1]		0.7	1.0
Steamroller	9.9	∞	6.3	∞	2246 14.5 [4]	1.5	0.18
$x \neq 0 \rightarrow x^2 > 0$	0.7	0.8/46 [3]	0.5	0.6/42 [3]		2.4	0.8
Bledsoe 1 `ANA/ANA003-4`	∞	7.4 [2]	∞	∞		∞	87
Bledsoe 2 `ANA/ANA004-4`	∞	32 [2]	∞	∞		∞	∞
Natnum3	2.1	0.15	0.6	0.05		0.07	0.03
Wos 4 `GRP/GRP008-1`	20	∞	43	∞		22	13
Pelletier 48	1.1	4.1	∞	∞		5.9	0.2

Remarks: 1 – With (back) factoring. 2 – With lemmas.
3 – Depends from selected literal. 4 – "nosave" flag cleared.

Fig. 3. Runtime results (in seconds) for various provers.
Entries such as `MSC/MSC006-1` refer to the respective TPTP-names [SSY94]. All examples were drawn from that problem library without modification.

In the parameter space *selection function* vs. *no selection function* we will not strictly prefer the one to the other. Note the extreme dependence on the "right" choice in the $x \neq 0 \rightarrow x^2 > 0$-example (this holds also for the Bledsoe examples), while in the Eder-example the right choice is not that crucial. On the other side, the Non-Obvious, Steamroller and Wos 4 examples obviously require the use of several contrapositives.

From this results we learn that the selection function should be carefully determined. Here, heuristics are conceivable such as "always select a biggest (in some ordering) head

literal" in order to work in a decreasing direction. The selection function can even be determined dynamically within the bounds of Definition 4.

Currently, the user has to supply the selection function for a given input clause. This function is inherited to all instances of that clause. Here we see potential for further improvements.

But even at the moment our restart provers can well compete with the MPRF prover, which is according to our classification, closest to the strict restart prover with selection function (see also the next section).

Ordinary model elimination as implemented by ME-PTTP and Setheo is sometimes faster than restart model elimination. In the case of Setheo this may especially be due to the numerous refinements not present in the other provers. However there are many examples where restart model elimination finds a proof more quickly, which suggest to us that it is an interesting alternative to traditional model elimination.

5 Related Work

Connection Method. The *connection method* [Bib87] is an analytic calculus closely related to model elimination. Clause sets are called *matrices* there, and a *path through a matrix* is obtained by taking exactly one literal from every clause in the matrix. The method proceeds by systematically checking all paths through the matrix to contain complementary literals. If this is the case, a refutation has been found.

A somewhat higher-level formulation of the connection method can be found in [Ede92], and in [BF93] we showed that this connection method can, in steps, simulate model elimination. The converse, however, is not true for the following essential difference between the connection method and model elimination: in model elimination in extension steps a complementary pair of literals (called *connection*) must be established between the *leaf* literal where the extension occurred and some literal of the extending clause. In the connection method this restriction is dropped, and so every literal along the path (or even none) may be part of the connection.

This property is also the key for the observation stated in the introduction, namely that the connection method is complete without the use of contrapositives. In order to see this, recall that a restart step consists of copying the first literal of the path, followed by an extension step. Thus, copying is not necessary if the first literal in the path is accessible for the connection — as is the case in the connection method. Hence we get as a corollary to theorem 7, the completeness of strict restart model elimination with selection function:

Corollary 10. *The connection method is complete for input sets in goal-normal form, even if no contrapositives are used.*

Problem Reduction Formats. In [Pla88] two calculi named *simplified problem reduction format* and *modified problem reduction format* are described. They are goal-oriented, and neither of these needs contrapositives. We will discuss both of them.

The *simplified problem reduction format* (SPRF) is a variant of the Gentzen sequent calculus (see e.g. [Gal87]). A sequent is pair, written as $\Gamma \rightarrow L$ where Γ is a list of

literals, and L is a literal. From the model elimination point of view a sequent $\Gamma \rightarrow L$ corresponds to the path $\Gamma \circ \langle \neg L \rangle$, i.e. the goal L is to be proven in the context (ancestor list) Γ.

Clauses are translated to inference rules operating on sequents; a clause $L \leftarrow L_1, \ldots, L_n$ is translated into the inference rule (where Γ is a variable)

$$\frac{\Gamma \rightarrow L_1 \qquad \cdots \qquad \Gamma \rightarrow L_n}{\Gamma \rightarrow L}$$

The interesting case is to see how model elimination restart steps can be mapped to derivations in SPRF. Suppose we have in a restart model elimination derivation a leaf $\neg p$ and wish to extend with the clause $p, q \leftarrow r$. After copying, the situation looks as depicted in Figure 2 (left side). This situation can be mirrored in SPRF by the partial proof in Figure2 (right side).

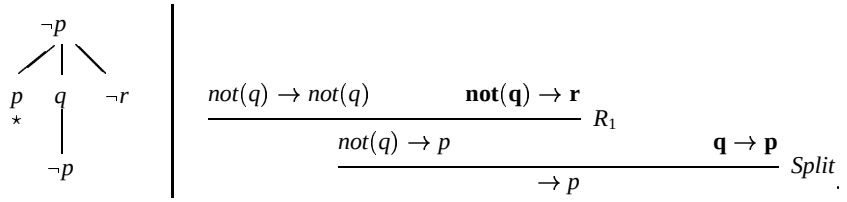


Fig. 2. Restart Model Elimination vs. Simplified Problem Reduction Format.

The *Split* rule is in effect the cut-rule, and R_1 stems from the clause $p, q \leftarrow r$. While the sequent $not(q) \rightarrow not(q)$ is an instance of an axiom, the boldface sequents are unproved. Note the close relationship to restart model elimination: the sequent $q \rightarrow p$ immediately corresponds to the goal $\neg p$ with ancestor list $\neg p \circ q$ in restart model elimination; it is even identical in strict restart model elimination, as negative ancestors need not be stored. For the other sequent $not(q) \rightarrow r$ note that the corresponding goal $\neg r$ in restart model elimination does not have the ancestor $not(q)$. If additional information – such as $not(q)$ – is considered as an advantage for proof finding, this is a shortcoming of restart model elimination. The situation however can easily be repaired either by an explicit change to the calculus, or by incorporating a more general *factoring* rule[3].

In this way, restart model elimination steps can be mapped to partial SPRF proofs. The converse, however, is not true. This is due to the fact that the splitting rule can be applied in every proof situation, i.e. to every sequent derived along a proof. In other words, a case analysis p or $\neg p$ can be carried even to goals totally unrelated to p.

Thus, in sum, the restart model elimination is more restricted than SPRF.

[3] Factoring means that a branch may be closed if its leaf is identical to some brother node of a predecessor of this leaf.

The *modified problem reduction format* (MPRF) avoids the problem of uncontrolled application of the splitting rule. This is formally carried out by an additional syntactical layer between "sequents" and "inference rules". As an essential difference, MPRF allows (in our terminology) for restart steps with *any* goal along the current path, not just with the goal literal as in restart model elimination. While this feature clearly increases the local search space, shorter proofs may be enabled. Another notable difference is that restart model elimination includes the negative literals along paths. As our experiments show this is often valuable information and should not be thrown away.

Near-Horn Prolog. As already mentioned in the introduction, there is a close relation to Loveland's Near-Horn Prolog, especially to the InH-Prolog variant from [LR89]. Instead of one tableau in our model elimination calculi, InH-Prolog deductions consist of a sequence of Prolog-like computations, called blocks. The activation of such blocks corresponds to our restart extension steps. If we agree that Prolog stepwisely transforms a goal set `G` into the empty goal set, then the Prolog-like computations in InH-Prolog deal with triples of the form `G # A { D }`. Here, the list `A` is called *active heads* and the list `D` is called *deferred heads*. These components can easily be explained from the viewpoint of restart model elimination: the active heads `A` corresponds to the *positive* literals of the path in restart model elimination which was most recently selected for a restart step; consequently, since `A` is a left-ended stack the leftmost literal in `A` is the literal which caused the restart step. In the Prolog-like restart blocks every literal in `A` may be used in the role of a unit input clause ("cancellation step) in order to get rid of a goal literal. The deferred heads `D` correspond to the remaining positive leaf literals of the path multiset; they will cause new restart blocks (or restart steps) at a later time.

Let us compare our refutation from Figure 1 (right side) with the following InH-Prolog refutation. In this example no deferred head occurs.

```
?- GOAL
:- P,Q
:- Q,Q          % factoring to simplify presentation!
:- Q
:-     # P      % P from disjunctive clause Q,P <-
                % is deferred, and the Block is finished
% restart:
?-GOAL # P
:- P,Q # P      % cancellation (reduction)
:- Q   # P
:- P   # P      % cancellation
:-     # P      %
```

The cancellation steps in this derivation correspond to the two reduction steps in the right subtree of Figure 1 (right side). The derivation from the left subtree does not have a counterpart in the above InH-Prolog refutation, because of the factoring step we performed in the first block; this, of course, would have been possible in the restart model elimination refutation.

The reduction steps starting from positive leaf-literals have no counterpart in InH-Prolog - within a block there are only extension or cancellation steps. The latter correspond to reductions with a negative leaf-literal.

On the other side, the concept of a *strong cancellation pruning rule* of InH-Prolog has (so far) no counterpart in restart model elimination. By this rule, a certain class of refutations is discarded. Stated positively, and in the terminology of restart model elimination, only those refutations are acceptable in which a literal which caused a restart step is used in a (any) subsequent reduction step. Thus restart steps not relevant for the proof are filtered out. The completeness of this restriction can be seen again by analyzing the completeness proof of Lemma 8. In brief, a restart step applied to $A, B \leftarrow C$ causes by the splitting rule Horn refutations with $A \leftarrow C$ and B. Now, if the given clause set is supposed (without loss of generality) to be *minimal* unsatisfiable, then also the splitted sets contain minimal unsatisfiable subsets containing $A \leftarrow C$ and B, respectively. Hence these clauses must be used in the Horn refutations, and consequently, the restart step occurring at B must be followed by a reduction step to B.

Summarizing on all these considerations we conclude that InH-Prolog is very closely related to strict restart model elimination without selection function. As a consequence we see that our PTTP implementation can be seen as an implementation of InH-Prolog.

SLWV-Resolution. In [PCA91] a theorem prover that retains the procedural aspects of logic programming is defined. This so called SLWV resolution system is based on SL-resolution, a linear resolution format. SLWV saves contrapositives and uses case analysis as an additional inference rule. To this end the usual resolution step from SL-resolution is modified such that besides the current goal any ancestor is allowed to be expanded. In our terminology this would mean that every negative literal along a path can be copied in a restart step. As the authors of [PCA91] explain, this freedom clearly increases the search space when compared to Near-Horn Prolog in the case of near-Horn problems. As a further difference to our restart model elimination, SLWV-Resolution needs a completely new framework. Pereira et.al. had to redesign the PTTP-implementation technique for their prover, whereas we were able to implement restart model elimination by a small change of our existing prover.

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